

Homework No.4, 550.695, Due March 26, 2009.

1. This problem discusses some simple, useful *transformations of PDFs*.

(a) Suppose that $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a differentiable, 1-to-1 and onto map and that $\tilde{\mathbf{X}}$ is an n -dimensional random vector with PDF $p_{\mathbf{X}}(\mathbf{x})$. If another n -dimensional random vector is defined by $\tilde{\mathbf{Y}} = \mathbf{f}(\tilde{\mathbf{X}})$, then show that its PDF is given by

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{p_{\mathbf{X}}(\mathbf{f}^{-1}(\mathbf{y}))}{\left| \det \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{f}^{-1}(\mathbf{y})) \right] \right|},$$

where \mathbf{f}^{-1} is the inverse map and $\partial \mathbf{f} / \partial \mathbf{x}$ is the Jacobian matrix.

Hint: Argue that $P(\tilde{\mathbf{Y}} \in B) = P(\tilde{\mathbf{X}} \in A)$ for $A = \mathbf{f}^{-1}(B)$ and use the change of variables formula from multivariate calculus.

Remark: The assumptions that \mathbf{f} is 1-1 and onto are not necessary. A more general result is the following:

$$p_{\mathbf{Y}}(\mathbf{y}) = \sum_{k=1}^{n(\mathbf{y})} \frac{p_{\mathbf{X}}(\mathbf{f}_k^{-1}(\mathbf{y}))}{\left| \det \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{f}_k^{-1}(\mathbf{y})) \right] \right|},$$

when there are $n(\mathbf{y})$ points $\mathbf{x}_k = \mathbf{f}_k^{-1}(\mathbf{y})$, $k = 1, \dots, n(\mathbf{y})$ that \mathbf{f} maps to \mathbf{y} and $p_{\mathbf{Y}}(\mathbf{y}) = 0$ when there are no such points.

(b) As a simple application of this result, consider a pair of random variables (\tilde{U}, \tilde{V}) with joint PDF $p_{U,V}(u, v)$. If sum and difference variables are defined by $\tilde{X} = \tilde{U} + \tilde{V}$ and $\tilde{Y} = \frac{1}{2}(\tilde{U} - \tilde{V})$, then show that (\tilde{X}, \tilde{Y}) has joint PDF

$$p_{X,Y}(x, y) = p_{U,V}\left(\frac{1}{2}x + y, \frac{1}{2}x - y\right)$$

and that the sum variable \tilde{X} has PDF

$$p_X(x) = \int_{-\infty}^{\infty} du p_{U,V}(u, x - u).$$

Hint: Use the fact that $p_X(x) = \int_{-\infty}^{\infty} dy p_{X,Y}(x, y)$.

2. This problem discusses some standard examples of random variables.

(a) A *Gamma*(a, b) random variable $\tilde{X} \sim \Gamma(a, b)$ for real a, b with $b > 0$ is a nonnegative random variable with the gamma PDF

$$p_{a,b}(x) = x^{a-1} \frac{b^a}{\Gamma(a)} e^{-bx}, \quad x \geq 0$$

and $p_{a,b}(x) = 0$ for $x < 0$. Show that the p th moment of this variable equals

$$E(\tilde{X}^p) = (a)^p / b^p,$$

where $(a)^p = a(a+1) \cdots (a+p-1)$ is the so-called *rising p th-factorial* of a .

Hint: Show that $\int_0^\infty x^{a-1} e^{-bx} dx = \Gamma(a)/b^a$. Then either differentiate both sides p times with respect to b or use the fact that $(a)^p = \Gamma(a+p)/\Gamma(a)$.

(b) The *Inverse-Gamma*(a, b) random variable $\tilde{X} \sim \Gamma^{-1}(a, b)$ for real a, b with $b > 0$ is a nonnegative random variable with the PDF

$$\bar{p}_{a,b}(x) = \frac{b^a}{\Gamma(a)} x^{-(1+a)} e^{-b/x}, \quad x \geq 0$$

and $\bar{p}_{a,b}(x) = 0$ for $x < 0$. Show that this is the density of the random variable $\tilde{X} = 1/\tilde{Y}$ where $\tilde{Y} \sim \Gamma(a, b)$.

(c) Show that a $\Gamma(\frac{1}{2}, \frac{1}{2})$ random variable \tilde{X} has the PDF

$$p_X(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \quad x > 0.$$

This is called a *chi-square random variable*. Show the square $\tilde{X} = \tilde{N}^2$ of a normal $N(0, 1)$ random variable \tilde{N} is chi-squared. Show that the CDF of a chi-square variable is given by $F_X(x) = 0$ for $x < 0$ and

$$F_X(x) = \operatorname{erf} \left(\sqrt{\frac{x}{2}} \right), \quad x > 0.$$

(d) Show that a $\Gamma^{-1}(\frac{1}{2}, \frac{1}{2})$ random variable \tilde{X} has the PDF

$$p_X(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-1/2x}, \quad x > 0.$$

This is called a *Lévy random variable*. Show the inverse-square $\tilde{X} = \tilde{N}^{-2}$ of a normal $N(0, 1)$ random variable is a Lévy variable. Show that the CDF of a Lévy variable is given by $F_X(x) = 0$ for $x < 0$ and

$$F_X(x) = \operatorname{erfc} \left(\sqrt{\frac{1}{2x}} \right), \quad x > 0.$$

3. This problem discusses *self-adjoint projection operators* on a Hilbert space \mathcal{H} .

(a) As in the class notes, for any $\Phi \in \mathcal{H}$ with $\|\Phi\| = 1$, we define

$$\hat{P}_\Phi \Psi = \langle \Phi, \Psi \rangle \Phi, \quad \forall \Psi \in \mathcal{H}.$$

Show that this operator satisfies the defining properties of a self-adjoint projection:

$$\hat{P}_\Phi^* = \hat{P}_\Phi, \quad \hat{P}_\Phi^2 = \hat{P}_\Phi.$$

Hint: $\hat{P}_\Phi^* = \hat{P}_\Phi$ is equivalent to $\langle \Psi_1, \hat{P}_\Phi \Psi_2 \rangle = \langle \hat{P}_\Phi \Psi_1, \Psi_2 \rangle$ for all $\Psi_1, \Psi_2 \in \mathcal{H}$.

(b) For any self-adjoint projection \hat{P} one defines the *orthogonal projection*

$$\hat{Q} = \hat{P}^\perp = \hat{I} - \hat{P},$$

where \hat{I} is the identity operator. Show that \hat{Q} satisfies

$$\hat{Q}^* = \hat{Q}, \quad \hat{Q}^2 = \hat{Q}, \quad \hat{P}\hat{Q} = \hat{Q}\hat{P} = 0.$$

(c) There is a one-to-one correspondence between self-adjoint projection operators \hat{P} on \mathcal{H} and closed linear subspaces $\mathcal{A} \subseteq \mathcal{H}$:

$$\mathcal{A} \longleftrightarrow \hat{P}_\mathcal{A}. \quad (*)$$

In quantum theory the subspaces $\mathcal{A} \subseteq \mathcal{H}$ are the analogues of the events $A \subset \Omega$ in classical probability, just as the projections $\hat{P}_\mathcal{A}$ are the analogues of the characteristic functions $\tilde{1}_A$. To see the connection (*), given a projection \hat{P} define the subspace

$$\mathcal{A} = \text{range}(\hat{P}) = \{\text{eigenspace of } \hat{P} \text{ for eigenvalue } 1\}$$

and the orthogonal subspace

$$\mathcal{A}^\perp = \{\Psi \in \mathcal{H} : \langle \Psi, \Phi \rangle = 0 \quad \forall \Phi \in \mathcal{A}\}.$$

Using the notations in (b), show that $\Psi = \hat{P}\Psi + \hat{Q}\Psi$ gives a decomposition of any Hilbert space vector Ψ into components

$$\hat{P}\Psi \in \mathcal{A}, \quad \hat{Q}\Psi \in \mathcal{A}^\perp$$

such that $\|\Psi\|^2 = \|\hat{P}\Psi\|^2 + \|\hat{Q}\Psi\|^2$.

(d) For the projection \hat{P}_Φ in (a), show that \mathcal{A} is the 1-dimensional subspace in Hilbert space spanned by the vector Φ , i.e. $\mathcal{A} = \{c\Phi, c \in \mathbb{C}\}$.