Homework No.4, 550.695, Due November 15, 2011.

1. This problem discusses some simple, useful *transformations of PDFs.*

(a) Suppose that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable, 1-to-1 and onto map and that $\tilde{\mathbf{X}}$ is an n-dimensional random vector with PDF $p_{\mathbf{X}}(\mathbf{x})$. If another n-dimensional random vector is defined by $\tilde{\mathbf{Y}} = \mathbf{f}(\tilde{\mathbf{X}})$, then show that its PDF is given by

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{p_{\mathbf{X}}\left(\mathbf{f}^{-1}(\mathbf{y})\right)}{\left|\det\left[\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\left(\mathbf{f}^{-1}(\mathbf{y})\right)\right]\right|},$$

where $\mathbf{f}^{-1}$ is the inverse map and $\partial\mathbf{f}/\partial\mathbf{x}$ is the Jacobian matrix.

Hint: Argue that $P(\tilde{\mathbf{Y}} \in B) = P(\tilde{\mathbf{X}} \in A)$ for $A = \mathbf{f}^{-1}(B)$ and use the change of variables formula from multivariate calculus.

Remark: The assumptions that $\mathbf{f}$ is 1-1 and onto are not necessary. A more general result is the following:

$$p_{\mathbf{Y}}(\mathbf{y}) = \sum_{k=1}^{n(\mathbf{y})} \frac{p_{\mathbf{X}}\left(\mathbf{f}_k^{-1}(\mathbf{y})\right)}{\left|\det\left[\frac{\partial\$\tilde{X}\sim\mathbf{f}}{\partial\mathbf{x}}\left(\mathbf{f}_k^{-1}(\mathbf{y})\right)\right]\right|},$$

when there are $n(\mathbf{y})$ points $\mathbf{x}_k = \mathbf{f}_k^{-1}(\mathbf{y}),\ k = 1, ..., n(\mathbf{y})$ that $\mathbf{f}$ maps to $\mathbf{y}$ and $p_{\mathbf{Y}}(\mathbf{y}) = 0$ when there are no such points.

(b) As a simple application of this result, consider a pair of random variables $(\tilde{U}, \tilde{V})$ with joint PDF $p_{U,V}(u, v)$. If sum and difference variables are defined by $\tilde{X} = \tilde{U} + \tilde{V}$ and $\tilde{Y} = \frac{1}{2}(\tilde{U} - \tilde{V})$, then show that $(\tilde{X}, \tilde{Y})$ has joint PDF

$$p_{X,Y}(x, y) = p_{U,V}(\frac{1}{2}x + y, \frac{1}{2}x - y)$$

and that the sum variable $\tilde{X}$ has PDF

$$p_X(x) = \int_{-\infty}^{\infty} du\, p_{U,V}(u, x - u).$$

Hint: Use the fact that $p_X(x) = \int_{-\infty}^{\infty} dy\, p_{X,Y}(x, y)$.

2. This problem discusses Gamma random variables and the Poisson point process. A *Gamma(a,λ) random variable* $\tilde{X} \sim \Gamma(a, \lambda)$ for real a, λ with $\lambda > 0$ is a nonnegative random variable with the gamma PDF

$$p_{a,\lambda}(x) = x^{a-1} \frac{\lambda^a}{\Gamma(a)} e^{-\lambda x}, \quad x \geq 0$$

and $p_{a,\lambda}(x) = 0$ for $x < 0$.

(a) Show that if $\tilde{U} \sim \Gamma(a, \lambda)$ and $\tilde{V} \sim \Gamma(b, \lambda)$ and if $\tilde{U}, \tilde{V}$ are *independent*, so that $p_{U,V}(u, v) = p_U(u) p_V(v)$, then $\tilde{X} = \tilde{U} + \tilde{V} \sim \Gamma(a + b, \lambda)$. *Hint:* You should look up the definition and properties of the Euler beta function $B(a, b)$.

(b) A special case of the gamma random variable *Gamma*(1,λ) with $a = 1$ is also called the *exponential random variable Exp*(λ). This random variable describes to a very good approximation the lifetime of an unstable quantum particle with decay rate λ. Show that the exponential random variable is "memoryless" in the following sense: if $\tilde{X} \sim Exp(\lambda)$ and if it is known that $\tilde{X} > c$, then $\tilde{Y} = \tilde{X} - c \sim Exp(\lambda)$. Prove this result by using the following definition of the conditional probability density:

$$p_Y(y|\tilde{X} > c) = \begin{cases} \frac{p_Y(y)}{P(\tilde{X} > c)} & y + c = x > c \\ 0 & y + c = x \leq c \end{cases}$$

(c) Consider a sample of radioactive material. Use parts (a) and (b) to show that the time $\tilde{T}_k$ of the decay of the kth unstable atom in the sample is distributed as a *Gamma*(k, λ) random variable. Assume that the decay time of each particle is statistically independent of all of the others.

(d) The *Poisson process* $\tilde{N}(t)$ is the random process which gives the total number of decays in a radioactive sample from $t = 0$ up to time $t \geq 0$. It can be expressed as

$$\tilde{N}(t) = \sum_{k=0}^{\infty} \theta(t - \tilde{T}_k)$$

in terms of the Heaviside step-function $\theta(t)$. Show that $\tilde{N}(t) = k$ if and only if $\tilde{T}_k \leq t$ and $\tilde{T}_{k+1} = \tilde{T}_k + \tilde{X}_k > t$. Use this result to explain why

$$P(\tilde{N}(t) = k) = \int_0^t d\tau \, P(\tilde{X} > t - \tau) p_{T_k}(\tau)$$

where $\tilde{X} \sim Exp(\lambda)$. Evaluate the integral to find $P(\tilde{N}(t) = k)$.

3. This problem discusses another of the simplest examples of a random process, the one-dimensional *Brownian motion* or *Wiener process*, $\tilde{W}(t)$. This is a Gaussian process, i.e. such that all of the random vectors $(\tilde{W}(t_1), \tilde{W}(t_2), ..., \tilde{W}(t_n))$ for any set of times $0 < t_1 < t_2 < \cdots < t_n$ have n-dimensional Gaussian distributions. It is furthermore defined by the three conditions

$$\tilde{W}(0) = 0, \quad \langle \tilde{W}(t) \rangle = 0, \quad \langle \tilde{W}(t)\tilde{W}(s) \rangle = t \wedge s,$$

for all times $t, s \geq 0$, where $t \wedge s = \min\{t, s\}$.

(a) Use the definition above to show that for any $t, s \geq 0$,

$$\langle |\tilde{W}(t) - \tilde{W}(s)|^2 \rangle = |t - s|,$$

and that for any set of times $t_2 > s_2 > t_1 > s_1$,

$$\langle [\tilde{W}(t_2) - \tilde{W}(s_2)][\tilde{W}(t_1) - \tilde{W}(s_1)] \rangle = 0.$$

The latter property shows that the increments $\tilde{W}(t) - \tilde{W}(s)$ for a time interval $[s, t]$ are uncorrelated for non-overlapping intervals.

(b) One way to obtain the Brownian motion is by a continuum limit of a random walk. The *random walk* is defined to be the discrete-time process with $\tilde{W}_0 = 0$ and

$$\tilde{W}_k = \sum_{j=1}^{k} \tilde{S}_j, \quad k \geq 1,$$

where

$$\tilde{S}_j = \begin{cases} +1 & \text{w/ probability } 1/2 \\ -1 & \text{w/ probability } 1/2 \end{cases}$$

and $\langle \tilde{S}_i \tilde{S}_j \rangle = 0$ for $i \neq j$. Thus, $\tilde{W}_k$ represents the position of a walker after k independent random steps to the right $(+1)$ or to the left (-1) with equal probability. For each time $t \geq 0$, define the continuous time process

$$\tilde{W}^\epsilon(t) = \epsilon \tilde{W}_{[\![t/\epsilon^2]\!]},$$

where $[\![x]\!]$ denotes the greatest integer $\leq x$. Thus $\tilde{W}^\epsilon(t)$ is the position of a random walker after approximately t/ϵ^2 steps of size ϵ. Show that

$$\tilde{W}^\epsilon(0) = 0, \quad \langle \tilde{W}^\epsilon(t) \rangle = 0, \quad \langle \tilde{W}^\epsilon(t)\tilde{W}^\epsilon(s) \rangle = \epsilon^2 [\![\tfrac{t \wedge s}{\epsilon^2}]\!],$$

for all times $t, s \geq 0$ and argue that $\epsilon^2 [\![\frac{t \wedge s}{\epsilon^2}]\!] \rightarrow t \wedge s$ for $\epsilon \rightarrow 0$. We shall see later that the Central Limit Theorem implies also that the limiting random process $\tilde{W}(t) = \lim_{\epsilon \to 0} \tilde{W}^\epsilon(t)$ is Gaussian and thus is in fact the Wiener process.

(c) Define the probability

$$p_{j,k} = \text{Prob}\left(\tilde{W}_k = j\right),$$

for the random-walker to be at integer position j at the discrete time k. Explain why

$$p_{j,k+1} = \frac{1}{2}p_{j+1,k} + \frac{1}{2}p_{j-1,k}. \tag{$*$}$$

Now define

$$p^\epsilon(x,t) = \frac{1}{\epsilon}\text{Prob}\left(x_\epsilon \le \tilde{W}^\epsilon(t) < x_\epsilon + \epsilon\right),$$

with $x_\epsilon = \epsilon[\![x/\epsilon]\!]$, so that $p(x,t) = \lim_{\epsilon\to 0} p^\epsilon(x,t)$ is the probability density function for the random variable $\tilde{W}(t)$. Use the result (*) to argue that

$$\partial_t p(x,t) = \frac{1}{2}\partial_x^2 p(x,t).$$