Homework #4 – Solutions

Problem 1. (a) If $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, onto and differentiable, then the change of variables formula of multivariate calculus states that, for any integrable function φ on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \varphi(\mathbf{y})\, d^n y = \int_{\mathbb{R}^n} \varphi(\mathbf{f}(\mathbf{x})) \left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right) \right| d^n x .$$

We apply this result to the function

$$\varphi(\mathbf{y}) = \tilde{1}_B(\mathbf{y})\, p_{\mathbf{Y}}(\mathbf{y})$$

to obtain

$$\int_B p_{\mathbf{Y}}(\mathbf{y})\, d^n y = \int_{\mathbb{R}^n} \tilde{1}_B(\mathbf{f}(\mathbf{x}))\, p_{\mathbf{Y}}(\mathbf{f}(\mathbf{x})) \left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right) \right| d^n x .$$

By definition of the density $p_{\mathbf{Y}}(\mathbf{y})$,

$$\begin{aligned} \int_B p_{\mathbf{Y}}(\mathbf{y})\, d^n y &= P(\tilde{\mathbf{Y}} \in B) \\ &= P(\mathbf{f}(\tilde{\mathbf{X}}) \in B) \\ &= P(\tilde{\mathbf{X}} \in \mathbf{f}^{-1}(B)) = P(\tilde{\mathbf{X}} \in A) \end{aligned}$$

Also,

$$\tilde{1}_B(\mathbf{f}(\mathbf{x})) = \tilde{1}_A(\mathbf{x})$$

since

$$\mathbf{f}(\mathbf{x}) \in B \iff \mathbf{x} \in \mathbf{f}^{-1}(B) = A .$$

We thus see that

$$P(\tilde{\mathbf{X}} \in A) = \int_A p_{\mathbf{Y}}(\mathbf{f}(\mathbf{x})) \left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right) \right| d^4x .$$

It therefore follows by the definition of the density that

$$p_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{Y}}(\mathbf{f}(\mathbf{x})) \left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right) \right| .$$

Substituting $\mathbf{y} = \mathbf{f}(\mathbf{x})$, this is equivalent to

$$p_{\mathbf{X}}(\mathbf{f}^{-1}(\mathbf{y})) = p_{\mathbf{Y}}(\mathbf{y}) \left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{f}^{-1}(\mathbf{y})) \right) \right|$$

or,

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{p_{\mathbf{X}}(\mathbf{f}^{-1}(\mathbf{y}))}{\left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{f}^{-1}(\mathbf{y})) \right) \right|} .$$

(b) We apply the previous result with

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u+v \\ \frac{1}{2}(u-v) \end{pmatrix} = \mathbf{f}(\mathbf{u})$$

for $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$. Note that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

so that

$$\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right) = (1)\left(-\tfrac{1}{2}\right) - (1)\left(\tfrac{1}{2}\right) = -1 .$$

Furthermore, solving the equations

$$u+v = x$$
$$u-v = 2y$$

for u & v gives

$$u = \tfrac{1}{2}x + y \ , \ v = \tfrac{1}{2}x - y$$

or

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + y \\ \frac{1}{2}x - y \end{pmatrix} = \mathbf{f}^{-1}(\mathbf{x}) .$$

We thus see that

$$p_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{U}}(\mathbf{f}^{-1}(\mathbf{x})) \Big/ \left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{f}^{-1}(\mathbf{x}))\right) \right|$$

or that

$$P_{X,Y}(x,y) = P_{U,V}(u,v) \Big/ |-1|$$

$$= P_{U,V}\left(\tfrac{1}{2}x + y, \tfrac{1}{2}x - y\right)$$

Since

$$P_X(x) = \int P_{X,Y}(x,y)\, dy$$

it follows that

$$P_X(x) = \int P_{U,V}\left(\tfrac{1}{2}x+y, \tfrac{1}{2}x-y\right) dy$$

$$= \int P_{U,V}(u, x-u)\, du$$

using $u = \frac{1}{2}x + y$ so that $du = dy$ (at fixed x) and

$$x - u = x - \left(\tfrac{1}{2}x + y\right) = \tfrac{1}{2}x - y.$$

<u>Remark</u>: If $\tilde{U}, \tilde{V}$ are independent random variables, notice that the sum variable $\tilde{X} = \tilde{U} + \tilde{V}$ has the PDF

$$P_X(x) = \int P_U(u)\, P_V(x-u)\, du$$

$$= (P_U * P_V)(x)$$

where "$*$" denotes the <u>convolution</u> product of functions.

Problem 2. (a) Using the definition of the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

we see by taking $z = a$ and $t = bx$, that

$$\Gamma(a) = b^a \int_0^\infty x^{a-1} e^{-bx}\, dx$$

or that

$$\int_0^\infty x^{a-1} e^{-bx}\, dx = \frac{\Gamma(a)}{b^a}.$$

We now take p ~~b~~-derivatives of both sides of this equation. First,

$$\frac{\partial^p}{\partial b^p}\left(\int_0^\infty x^{a-1} e^{-bx}\, dx \right) = \int_0^\infty x^{a-1} (-x)^p e^{-bx}\, dx$$

$$= (-1)^p\, E(\tilde{X}^p) \cdot \frac{\Gamma(a)}{b^p}$$

On the other hand,

$$\frac{\partial^p}{\partial b^p}\left(\frac{\Gamma(a)}{b^a} \right) = (-a)(-a-1)\cdots(-a-p+1)\, \frac{\Gamma(a)}{b^{a+p}}$$

$$= (-1)^p \frac{(a)^p}{b^p} \cdot \frac{\Gamma(a)}{b^p}$$

so that

$$E(\tilde{X}^p) = (a)^p / b^p.$$

Another approach is to note that

$$E(\tilde{X}^p) = \frac{b^a}{\Gamma(a)} \int_0^\infty x^p \cdot x^{a-1} e^{-bx} dx$$

$$= \frac{b^a}{\Gamma(a)} \int_0^\infty x^{a+p-1} e^{-bx} dx$$

$$= \frac{b^a}{\Gamma(a)} \cdot \frac{\Gamma(a+p)}{b^{a+p}} = \frac{\Gamma(a+p)/\Gamma(a)}{b^p} .$$

A basic property of the gamma function is that

$$\Gamma(z+1) = z\Gamma(z) .$$

Applying this p times gives

$$\Gamma(a+p) = (a+p-1)(a+p-2) \cdots a\, \Gamma(a)$$

or

$$\frac{\Gamma(a+p)}{\Gamma(a)} = a(a+1) \cdots (a+p-1)$$

$$= (a)^p ,$$

so that

$$E(\tilde{X}^p) = \frac{(a)^p}{b^p} . \checkmark$$

b) We use the change of variables formula with

$$x = g(y) = 1/y$$

where

$$p_Y(y) = y^{a-1} \frac{b^a}{\Gamma(a)} e^{-by}, \quad y > 0.$$

Since

$$\frac{dx}{dy} = -\frac{1}{y^2},$$

we get from the change of variables formula that

$$p_X(x) = \frac{p_Y(y)}{|dx/dy|} = y^2 p_Y(y) \Big|_{y=1/x}$$

$$= \frac{1}{x^2} \cdot \left(\frac{1}{x}\right)^{a-1} \frac{b^a}{\Gamma(a)} e^{-b/x}$$

$$= \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{-b/x}.$$

c) Taking $a = \frac{1}{2}$, $b = \frac{1}{2}$ in the gamma density gives

$$p(x) = p_{1/2, 1/2}(x) = x^{\frac{1}{2}-1} \frac{(1/2)^{1/2}}{\Gamma(\frac{1}{2})} e^{-x/2}.$$

We must furthermore use the result that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$

so that

$$p(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \quad x > 0$$

We next show that $\tilde{X} = \tilde{N}^2$ with $\tilde{N} \sim N(0,1)$ is chi-squared.
Taking $x = n^2$, we see that the inverse is <u>double-valued</u>:

$$n_\pm = \pm\sqrt{x},$$

with

$$\frac{dn_\pm}{dx} = \frac{\pm 1}{2} x^{-1/2}.$$

Thus, we must use the generalized change of variables formula

$$p_X(x) = p_N(n_+) \left|\frac{dn_+}{dx}\right| + p_N(n_-) \left|\frac{dn_-}{dx}\right|$$

with

$$p_N(n) = \frac{1}{\sqrt{2\pi}} e^{-n^2/2}.$$

This gives

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{2} x^{-1/2} + \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2},$$

which is the chi-squared PDF.

Since $\tilde{X} > 0$ with probability one,

$$F_{\mathbb{X}}(x) = P(X < x) = 0$$

for any $x < 0$, For $x > 0$ we get that

$$F_{\mathbb{X}}(x) = \int_0^x du \frac{1}{\sqrt{2\pi u}} e^{-u/2} .$$

Then set

$$u = 2t^2$$

$$\sqrt{2u} = 2t$$

$$du = 4t\,dt$$

giving

$$F_{\mathbb{X}}(x) = \int_0^{\sqrt{x/2}} 4t\,dt \cdot \frac{1}{2t\sqrt{\pi}} e^{-t^2}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x/2}} dt\, e^{-t^2} = \mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)$$

(d) Setting $a = \frac{1}{2}, b = \frac{1}{2}$ in the inverse gamma PDF gives

$$P_{1/2,1/2}(x) = \frac{(1/2)^{1/2}}{\Gamma(\frac{1}{2})} \cdot x^{-(1+\frac{1}{2})} e^{-1/2x}$$

$$= \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-1/2x} , x > 0$$

From part (c), if $\tilde{N} \sim N(0,1)$, then

$$\tilde{N}^2 \sim \chi\text{-square} \sim \Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

Thus,

$$\tilde{X} = \tilde{N}^{-2} \sim \Gamma^{-1}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

To calculate the CDF we note that $\tilde{Y} = 1/\tilde{X}$ is a chi-squared random variable, whose CDF was calculated in part (c). We see that

$$\begin{aligned}
F_{\tilde{X}}(x) &= P(\tilde{X} < x) \\
&= P\left(\frac{1}{\tilde{Y}} < x\right) \\
&= P\left(\tilde{Y} > \frac{1}{x}\right) \\
&= 1 - P\left(\tilde{Y} < \frac{1}{x}\right) \\
&= 1 - F_{\tilde{Y}}\left(\frac{1}{x}\right) \\
&= 1 - \operatorname{erf}\left(\sqrt{\frac{1/x}{2}}\right) \\
&= 1 - \operatorname{erf}\left(\sqrt{\frac{1}{2x}}\right) = \operatorname{erfc}\left(\sqrt{\frac{1}{2x}}\right)
\end{aligned}$$

by the definition of the complementary error function.

Problem 3. (a) For any $\Psi_1, \Psi_2 \in \mathcal{H}$,

$$\langle \Psi_1, \hat{P}_\Phi \Psi_2 \rangle = \langle \Psi_1, \langle \Phi, \Psi_2 \rangle \Phi \rangle$$

$$= \langle \Psi_1, \Phi \rangle \langle \Phi, \Psi_2 \rangle$$

$$= \overline{\langle \Phi, \Psi_1 \rangle} \langle \Phi, \Psi_2 \rangle$$

$$= \langle \langle \Phi, \Psi_1 \rangle \Phi, \Psi_2 \rangle$$

$$= \langle \hat{P}_\Phi \Psi_1, \Psi_2 \rangle$$

which is equivalent to

$$\hat{P}_\Phi^* = \hat{P}_\Phi .$$

On the other hand, for all $\Psi \in \mathcal{H}$

$$\hat{P}_\Phi^2 \Psi = \hat{P}_\Phi (\hat{P}_\Phi \Psi)$$

$$= \hat{P}_\Phi \left(\langle \Phi, \Psi \rangle \Phi \right)$$

$$= \langle \Phi, \langle \Phi, \Psi \rangle \Phi \rangle \Phi$$

$$= \langle \Phi, \Psi \rangle \underbrace{\langle \Phi, \Phi \rangle}_{= \|\Phi\|^2 = 1 \ !} \Phi$$

$$= \langle \Phi, \Psi \rangle \Phi = \hat{P}_\Phi \Psi$$

$$\Longrightarrow \quad \hat{P}_\Phi^2 = \hat{P}_\Phi$$

(b) Since $\hat{I}^* = \hat{I}$ (easily), we see that

$$\hat{Q}^* = (\hat{I} - \hat{P})^*$$
$$= \hat{I}^* - \hat{P}^* = \hat{I} - \hat{P} = \hat{Q} \checkmark$$

Also,

$$\hat{Q}^2 = (\hat{I} - \hat{P})^2 = \hat{I}^2 - 2\hat{I}\hat{P} + \hat{P}^2$$
$$= \hat{I} - 2\hat{P} + \hat{P} \qquad (\text{using } \hat{P}^2 = \hat{P})$$
$$= \hat{I} - \hat{P} = \hat{Q} \checkmark$$

Furthermore,

$$\hat{P}\hat{Q} = \hat{P}(\hat{I} - \hat{P})$$
$$= \hat{P} - \hat{P}^2 = \hat{P} - \hat{P} = 0 \checkmark$$

and, likewise,

$$\hat{Q}\hat{P} = (\hat{I} - \hat{P})\hat{P}$$
$$= \hat{P} - \hat{P}^2 = \hat{P} - \hat{P} = 0 \checkmark$$

NOTE: This last property implies that, for any $\Psi_1, \Psi_2 \in \mathcal{H}$,

$$\langle \hat{Q}\Psi_1, \hat{P}\Psi_2 \rangle = \langle \Psi_1, \hat{Q}^* \hat{P} \Psi_2 \rangle$$
$$= \langle \Psi_1, \hat{Q}\hat{P}\Psi_2 \rangle = 0$$

so that $\hat{Q}\Psi_1 \perp \hat{P}\Psi_2$. This explains the notation $\hat{Q} = \hat{P}^\perp$.

(c) From the definition of $\hat{Q}$,

$$\hat{I} = \hat{P} + \hat{Q}.$$

Thus,

$$\Psi = \hat{I}\Psi = (\hat{P}+\hat{Q})\Psi = \hat{P}\Psi + \hat{Q}\Psi$$

Since $\mathcal{A} = \text{range}(\hat{P})$, clearly $\hat{P}\Psi \in \mathcal{A}$!

Also, $\hat{P}\Psi$ is an eigenvector of $\hat{P}$ with eigenvalue $= 1$, since $\hat{P}(\hat{P}\Psi) = \hat{P}^2\Psi = \hat{P}\Psi = +1\cdot(\hat{P}\Psi)$.

Next we note that any $\Phi \in \mathcal{A}$ can be written, by definition as $\Phi = \hat{P}\Phi_0$ for some $\Phi_0 \in \mathcal{H}$. Then, $\forall \Phi \in \mathcal{A}$

$$\begin{aligned}\langle \hat{Q}\Psi, \Phi\rangle &= \langle \hat{Q}\Psi, \hat{P}\Phi_0\rangle \\ &= \langle \Psi, \hat{Q}^*\hat{P}\Phi_0\rangle \\ &= \langle \Psi, \hat{Q}\hat{P}\Phi_0\rangle = 0\end{aligned}$$

Thus, $\hat{Q}\Psi \in \mathcal{A}^\perp$, by definition.

Finally,
$$\begin{aligned}\|\Psi\|^2 &= \langle \Psi, \Psi\rangle \\ &= \langle \hat{P}\Psi + \hat{Q}\Psi, \hat{P}\Psi + \hat{Q}\Psi\rangle \\ &= \langle \hat{P}\Psi, \hat{P}\Psi\rangle + \underbrace{\langle \hat{Q}\Psi, \hat{P}\Psi\rangle}_{=0} \\ &\quad + \underbrace{\langle \hat{P}\Psi, \hat{Q}\Psi\rangle}_{=0} + \langle \hat{Q}\Psi, \hat{Q}\Psi\rangle \\ &= \|\hat{P}\Psi\|^2 + \|\hat{Q}\Psi\|^2.\end{aligned}$$

(d) By definition

$$\mathcal{A} = \text{range}(\hat{P})$$
$$= \{ \hat{P}\Psi \mid \Psi \in \mathcal{H} \}$$

For the projections in part (a),

$$\hat{P}\Psi = \langle \Phi, \Psi \rangle \Phi = c\,\Phi$$

with $c = \langle \Phi, \Psi \rangle \in \mathbb{C}$. Thus, every element of $\mathcal{A}$ has the required form. To complete the proof, we must only show that <u>any</u> value $c \in \mathbb{C}$ is allowed. But note that, for any choice of c,

$$\hat{P}(c\Phi) = c\,\hat{P}\Phi$$
$$= c\,\underbrace{\langle \Phi, \Phi \rangle}_{1}\,\Phi$$
$$= c\,\Phi$$

and thus

$$c\Phi \in \mathcal{A}.$$

We have thus shown that $\Psi \in \mathcal{A} = \text{range}(\hat{P})$ if and only if $\Psi = c\,\Phi$ for some $c \in \mathbb{C}$.