

Homework No.3, 550.695, Due March 5, 2009.

1. This problem discusses the Euler equations of a simple compressible fluid,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\partial_t \mathbf{j} + \nabla \cdot (p \mathbf{1} + \rho \mathbf{u} \mathbf{u}) = \mathbf{0}, \quad (2)$$

$$\partial_t e + \nabla \cdot [(e + p) \mathbf{u}] = 0, \quad (3)$$

where ρ is mass density, $\mathbf{j} = \rho \mathbf{u}$ is momentum density, e is the total energy density, and $p(\rho, e_0)$ is the thermodynamic pressure.

(a) Show that equation (1) can be rewritten as

$$D_t \rho + \rho(\nabla \cdot \mathbf{u}) = 0,$$

where $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the convective or material derivative.

(b) Show that equation (2) can be written in two alternative forms:

$$D_t \mathbf{j} + \mathbf{j}(\nabla \cdot \mathbf{u}) = -\nabla p,$$

or

$$D_t \mathbf{u} = -(\nabla p)/\rho.$$

(c) Use the results in part (b) to derive the balance equation for kinetic energy

$$\partial_t \left(\frac{1}{2} \rho u^2 \right) + \nabla \cdot \left[\left(\frac{1}{2} \rho u^2 + p \right) \mathbf{u} \right] = p(\nabla \cdot \mathbf{u}).$$

Interpret the righthand side in terms of work done by the pressure during fluid compression. Use the above result together with equation (3) to derive the equation for the internal energy e_0

$$\partial_t e_0 + \nabla \cdot (e_0 \mathbf{u}) = -p(\nabla \cdot \mathbf{u}),$$

where $e = e_0 + \frac{1}{2} \rho u^2$.

2. This problem introduces the *Onsager form* of the Navier-Stokes equation of a simple compressible fluid. Let ρ_μ for $\mu = 0, 1, \dots, 4$ denote the conserved densities

$$\rho_0 = \rho, \quad \rho_i = j_i \quad (i = 1, 2, 3), \quad \rho_4 = e.$$

The conjugate thermodynamic potentials $\lambda_\mu = -\partial s / \partial \rho_\mu$, where s is entropy, are

$$\lambda_0 = \frac{1}{T} \left(\mu - \frac{1}{2} u^2 \right), \quad \lambda_i = \frac{u_i}{T} \quad (i = 1, 2, 3), \quad \lambda_4 = -\frac{1}{T}.$$

In this problem we shall show that the dissipative fluxes $\mathbf{J}_\mu^{(1)}$ in the Navier-Stokes equation can be written in the form

$$J_{i\mu}^{(1)} = -L_{i\mu, j\nu} \partial_j \lambda_\nu,$$

where $L_{i\mu, j\nu}$ are the *Onsager coefficients* that are given for a simple fluid by

$$\begin{aligned} L_{i0, j\nu} &= 0, \\ L_{ik, jl} &= \eta T \left(\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj} - \frac{2}{3} \delta_{ik} \delta_{jl} \right) + \zeta T \delta_{ik} \delta_{jl}, \\ L_{ik, j4} &= \eta T \left(u_i \delta_{kj} + u_k \delta_{ij} - \frac{2}{3} \delta_{ik} u_j \right) + \zeta T \delta_{ik} u_j, \\ L_{i4, j4} &= (\kappa T + \eta u^2) T \delta_{ij} + \left(\zeta + \frac{1}{3} \eta \right) T u_i u_j, \end{aligned}$$

where Roman indices run from 1 – 3 and Greek indices run from 0 – 4, and where the symmetry condition $L_{i\mu, j\nu} = L_{j\nu, i\mu}$ holds. In particular, show that

$$\begin{aligned} (a) \quad & J_{i0}^{(1)} = -L_{i0, j\nu} \partial_j \lambda_\nu = 0, \\ (b) \quad & T_{ik}^{(1)} = J_{ik}^{(1)} = -L_{ik, j\nu} \partial_j \lambda_\nu, \\ (c) \quad & s_i^{(1)} + T_{ij}^{(1)} u_j = J_{i4}^{(1)} = -L_{i4, j\nu} \partial_j \lambda_\nu. \end{aligned}$$

Remark: The thermodynamic entropy production for the Navier-Stokes system can be written as

$$\sigma(t) = \int d^3r \, L_{i\mu, j\nu}(\rho(\mathbf{r}, t)) \partial_i \lambda_\mu(\mathbf{r}, t) \partial_j \lambda_\nu(\mathbf{r}, t).$$

Onsager's Principle states that the probability of a molecular fluctuation of the fluid system is given by the excess entropy production $\sigma_{ex}(t)$ required to produce the fluctuation, as

$$\text{Prob}[\rho] \sim \exp \left(-\frac{1}{4k_B} \int_{-\infty}^{+\infty} dt \, \sigma_{ex}(t) \right).$$

3. This problem develops a proof of the following theorem: if $F(x, \xi)$ is a bounded function that is smooth in x and period-1 in ξ , then

$$(*) \quad \lim_{\epsilon \rightarrow 0} \int_a^b dx F(x, x/\epsilon) = \int_a^b dx \bar{F}(x),$$

where $\bar{F}(x) = \int_0^1 d\xi F(x, \xi)$.

(a) Show that the definite integral

$$G(x, \xi) = \int_0^\xi d\eta (F(x, \eta) - \bar{F}(x))$$

defines a function which is also bounded, period-1 in ξ , and smooth in x . In particular, the partial derivative of G with respect to x is given by

$$G_x(x, \xi) = \int_0^\xi d\eta (F_x(x, \eta) - \bar{F}_x(x)).$$

(b) Use the fundamental theorem of calculus for $G(x, x/\epsilon)$ to show that

$$\int_a^b dx (F(x, x/\epsilon) - \bar{F}(x)) = \epsilon \left[G(b, b/\epsilon) - G(a, a/\epsilon) - \int_a^b dx G_x(x, x/\epsilon) \right].$$

Apply this result and part (a) to conclude the limit result (*).

(c) Define a gradient-length L_∇ for the mean-field by

$$L_\nabla = \min_x \left| \frac{\bar{F}(x)}{\bar{F}_x(x)} \right|.$$

Use the theorem (*) to argue that a coarse-grained field equals the mean-field,

$$\frac{1}{2\ell} \int_{x-\ell}^{x+\ell} dy F(y, y/\epsilon) \doteq \bar{F}(x),$$

for $\epsilon \ll \ell \ll L_\nabla$.