

## Homework No.3, 550.695, Due October 18, 2011

1. This problem discusses the two-scale diffusion problem for  $\epsilon \ll 1$

$$u_t(x, t) = (D(x/\epsilon)u_x(x, t))_x \quad (*)$$

with  $D(\xi) = [1 + \cos^2(2\pi\xi)]^{-1}$  and  $u_0(x) = \cos(2\pi x)$

(a) Calculate explicitly  $\bar{D}_H$ ,  $W(\xi)$ ,  $v_0(x, t)$ ,  $v_1(x, \xi, t)$  and thus the 1st-order approximation to the true solution

$$u(x, t) \sim u^{(1)}(x, t) \equiv v_0(x, t) + \epsilon v_1(x, x/\epsilon, t).$$

(b) Calculate the coarse-grained (approximate) solution

$$\bar{u}_\ell^{(1)}(x, t) = \frac{1}{2\ell} \int_{-\ell}^{\ell} d\rho u^{(1)}(x + \rho, t).$$

Useful integrals for this purpose are  $\frac{1}{2\ell} \int_{-\ell}^{\ell} d\rho \cos(k\rho) = \frac{\sin(k\ell)}{k\ell}$ ,  $\frac{1}{2\ell} \int_{-\ell}^{\ell} d\rho \sin(k\rho) = 0$ . Use your result to show that

$$\bar{u}_\ell^{(1)}(x, t) = v_0(x, t) \left[ 1 + O\left(\left(\frac{\ell}{L_\nabla}\right)^2\right) \right] + O\left(\frac{\epsilon^2}{\ell L_\nabla}\right)$$

and thus  $\bar{u}_\ell^{(1)}(x, t) \simeq v_0(x, t)$  for  $\epsilon \ll \ell \ll L_\nabla$  with  $L_\nabla^2 \equiv \min_x |u_0(x)/u_0''(x)|$ .

(c) The PDE (\*) can be converted to a (large) system of ODE's by discretizing the differential operator as

$$\frac{d}{dt}u(x, t) = \frac{1}{h^2} \left[ D\left(\frac{x+h}{\epsilon}\right) (u(x+h, t) - u(x, t)) + D\left(\frac{x}{\epsilon}\right) (u(x-h, t) - u(x, t)) \right]$$

for some  $h = 1/N$  and considering the vector  $\mathbf{u}(t) = (u(x_0, t), u(x_1, t), \dots, u(x_{N-1}, t))$  with  $x_i = ih$ ,  $i = 0, 1, \dots, N-1$ . If the resulting equation for  $\mathbf{u}(t)$  is solved by a standard integration method for ODE's, then this is called the "method of lines" for solving the original PDE. Use this method to approximate the solution to the problem in (a) for  $\epsilon = 1/100$  over the time interval  $0 < t < 0.1$ , with  $N = 1000$ . Note that the ODE for  $\mathbf{u}(t)$  is very stiff, so it is advisable to use an implicit method (like `ode15s` in MATLAB) especially designed for stiff problems. Compare the numerical approximation together with the asymptotic approximation from (a) by plotting them both for several times in the range  $0 < t < 0.1$ .

2. This problem discusses the Euler equations of a simple compressible fluid,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\partial_t \mathbf{j} + \nabla \cdot (p \mathbf{1} + \rho \mathbf{u} \mathbf{u}) = \mathbf{0}, \quad (2)$$

$$\partial_t e + \nabla \cdot [(e + p) \mathbf{u}] = 0, \quad (3)$$

where  $\rho$  is mass density,  $\mathbf{j} = \rho \mathbf{u}$  is momentum density,  $e$  is the total energy density, and  $p(\rho, e_0)$  is the thermodynamic pressure.

(a) Show that equation (1) can be rewritten as

$$D_t \rho + \rho (\nabla \cdot \mathbf{u}) = 0,$$

where  $D_t = \partial_t + \mathbf{u} \cdot \nabla$  is the convective or material derivative.

(b) Show that equation (2) can be written in two alternative forms:

$$D_t \mathbf{j} + \mathbf{j} (\nabla \cdot \mathbf{u}) = -\nabla p,$$

or

$$D_t \mathbf{u} = -(\nabla p) / \rho.$$

(c) Use the results in part (b) to derive the balance equation for kinetic energy

$$\partial_t \left( \frac{1}{2} \rho u^2 \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho u^2 + p \right) \mathbf{u} \right] = p (\nabla \cdot \mathbf{u}).$$

Interpret the righthand side in terms of work done by the pressure during fluid compression. Use the above result together with equation (3) to derive the equation for the internal energy  $e_0$

$$\partial_t e_0 + \nabla \cdot (e_0 \mathbf{u}) = -p (\nabla \cdot \mathbf{u}),$$

where  $e = e_0 + \frac{1}{2} \rho u^2$ .

3. This problem introduces the *Onsager form* of the Navier-Stokes equation of a simple compressible fluid. Let  $\rho_\mu$  for  $\mu = 0, 1, \dots, 4$  denote the conserved densities

$$\rho_0 = \rho, \quad \rho_i = j_i \quad (i = 1, 2, 3), \quad \rho_4 = e.$$

The conjugate thermodynamic potentials  $\lambda_\mu = -\partial s / \partial \rho_\mu$ , where  $s$  is entropy, are

$$\lambda_0 = \frac{1}{T} \left( \mu - \frac{1}{2} u^2 \right), \quad \lambda_i = \frac{u_i}{T} \quad (i = 1, 2, 3), \quad \lambda_4 = -\frac{1}{T}.$$

In this problem we shall show that the dissipative fluxes  $\mathbf{J}_\mu^{(1)}$  in the Navier-Stokes equation can be written in the form

$$J_{i\mu}^{(1)} = -L_{i\mu, j\nu} \partial_j \lambda_\nu,$$

where  $L_{i\mu,j\nu}$  are the *Onsager coefficients* that are given for a simple fluid by

$$L_{i0,j\nu} = 0,$$

$$L_{ik,jl} = \eta T \left( \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - \frac{2}{3}\delta_{ik}\delta_{jl} \right) + \zeta T \delta_{ik}\delta_{jl},$$

$$L_{ik,j4} = \eta T \left( u_i\delta_{kj} + u_k\delta_{ij} - \frac{2}{3}\delta_{ik}u_j \right) + \zeta T \delta_{ik}u_j,$$

$$L_{i4,j4} = (\kappa T + \eta u^2) T \delta_{ij} + \left( \zeta + \frac{1}{3}\eta \right) T u_i u_j,$$

where Roman indices run from 1 – 3 and Greek indices run from 0 – 4, and where the symmetry condition  $L_{i\mu,j\nu} = L_{j\nu,i\mu}$  holds. In particular, show that

$$\begin{aligned} (a) \quad & J_{i0}^{(1)} = -L_{i0,j\nu}\partial_j\lambda_\nu = 0, \\ (b) \quad & T_{ik}^{(1)} = J_{ik}^{(1)} = -L_{ik,j\nu}\partial_j\lambda_\nu, \\ (c) \quad & s_i^{(1)} + T_{ij}^{(1)}u_j = J_{i4}^{(1)} = -L_{i4,j\nu}\partial_j\lambda_\nu. \end{aligned}$$

**Remark:** The thermodynamic entropy production for the Navier-Stokes system can be written as

$$\sigma(t) = \int d^3r L_{i\mu,j\nu}(\rho(\mathbf{r}, t)) \partial_i \lambda_\mu(\mathbf{r}, t) \partial_j \lambda_\nu(\mathbf{r}, t).$$

*Onsager's Principle* states that the probability of a molecular fluctuation of the fluid system is given by the excess entropy production  $\sigma_{ex}(t)$  required to produce the fluctuation, as

$$\text{Prob}[\rho] \sim \exp \left( -\frac{1}{4k_B} \int_{-\infty}^{+\infty} dt \sigma_{ex}(t) \right).$$