Homework #3 – Solutions

Problem 1. (a) By the product rule

$$\nabla\cdot(u\rho) = (\nabla\cdot u)\rho + (u\cdot\nabla)\rho .$$

Thus,

$$0 = \partial_t\rho + \nabla\cdot(u\rho) = \partial_t\rho + (u\cdot\nabla)\rho + \rho(\nabla\cdot u)$$
$$= D_t\rho + \rho(\nabla\cdot u) \checkmark$$

using the definition $D_t\rho = \partial_t\rho + (u\cdot\nabla)\rho$.

(b) In the same manner, the product rule and $\mathbf{j} = \rho u$ gives

$$\nabla\cdot(\rho u u) = \nabla\cdot(\mathbf{j}u) = (u\cdot\nabla)\mathbf{j} + \mathbf{j}(\nabla\cdot u).$$

Also,
$$\nabla\cdot(p\mathbb{1}) = \nabla p$$

for the identity matrix $\mathbb{1}$. Thus,

$$0 = \partial_t\mathbf{j} + \nabla\cdot(p\mathbb{1} + \rho u u)$$
$$= \partial_t\mathbf{j} + (u\cdot\nabla)\mathbf{j} + \mathbf{j}(\nabla\cdot u) + \nabla p$$

or,
$$D_t\mathbf{j} + \mathbf{j}(\nabla\cdot u) = -\nabla p \checkmark$$

Noting again that $\mathbf{j} = \rho\mathbf{u}$ and using the chain rule

$$D_t \mathbf{j} = (D_t \rho)\mathbf{u} + \rho(D_t\mathbf{u})$$

$$= -\rho(\nabla\cdot\mathbf{u})\cdot\mathbf{u} + \rho(D_t\mathbf{u}) \qquad \text{by part (a)}$$

$$= -\mathbf{j}(\nabla\cdot\mathbf{u}) + \rho(D_t\mathbf{u})$$

or

$$\rho(D_t\mathbf{u}) = D_t\mathbf{j} + \mathbf{j}(\nabla\cdot\mathbf{u}) = -\nabla p$$

so that

$$D_t\mathbf{u} = -\frac{\nabla p}{\rho}$$

(c) We may write the kinetic energy density as $\frac{1}{2}\rho u^2 = \frac{1}{2}\mathbf{j}\cdot\mathbf{u}$ and then apply the product rule:

$$D_t\left(\tfrac{1}{2}\rho u^2\right) = D_t\left(\tfrac{1}{2}\mathbf{j}\cdot\mathbf{u}\right)$$

$$= -\tfrac{1}{2}\left[\nabla p + \mathbf{j}(\nabla\cdot\mathbf{u})\right]\cdot\mathbf{u} - \tfrac{1}{2}\mathbf{j}\cdot\frac{\nabla p}{\rho} \qquad \text{by (b)}$$

$$= -(\mathbf{u}\cdot\nabla)p - \tfrac{1}{2}\rho u^2(\nabla\cdot\mathbf{u}) \qquad \text{using } \mathbf{j} = \rho\mathbf{u}$$

Using the definition of the material derivative and the product rule, this may be rewritten as

$$\partial_t\left(\tfrac{1}{2}\rho u^2\right) + \mathbf{u}\cdot\nabla\left(\tfrac{1}{2}\rho u^2\right) + \tfrac{1}{2}\rho u^2(\nabla\cdot\mathbf{u}) = -\nabla\cdot(p\mathbf{u}) + p(\nabla\cdot\mathbf{u})$$

or

$$\partial_t\left(\tfrac{1}{2}\rho u^2\right) + \nabla\cdot\left(\left(\tfrac{1}{2}\rho u^2 + p\right)\mathbf{u}\right) = p(\nabla\cdot\mathbf{u})$$

Let us define the "specific volume", or volume per particle, as the inverse of the number density n:

$$v = 1/n = m/\rho .$$

Then,

$$\frac{D_t v}{v} = -\frac{D_t \rho}{\rho} = \nabla \cdot u$$

so that

$$p(\nabla \cdot u) = p D_t v \cdot n .$$

This expression has a simple thermodynamic meaning. Recall that pressure-volume work or PV-work is given by

$$dW = -P\, dV$$

in elementary thermodynamics, if a change in volume dV occurs at pressure P. Thus, $p D_t v$ is the negative of the pressure-volume work per particle performed by the motion of the fluid in a comoving (material) reference frame per unit time. Multiplied by the number density n, this gives the rate of loss of kinetic energy per unit time and per unit volume, due to the compressional work performed by the fluid motion.

The Euler equation for the total energy density is

$$\partial_t \left(e_0 + \tfrac{1}{2}\varrho u^2 \right) + \nabla \cdot \left[\left(h + \tfrac{1}{2}\varrho u^2 \right) \mathbf{u} \right] = 0$$

where e_0 is internal energy and $h = e_0 + p$ is enthalpy. Subtracting the balance equation derived above for the kinetic energy density, we obtain

$$\partial_t e_0 + \nabla \cdot (e_0 \mathbf{u}) = - p(\nabla \cdot \mathbf{u}).$$

Notice that the pressure-work now enters the balance equation for the internal energy density as a source term: whatever energy is lost by the fluid motion kinetic energy due to compressional work reappears as internal (heat) energy of the fluid. This exchange is entirely reversible and can go in both directions.

For example, the compressible Euler equations support sound waves, in which the energy density at a point oscillates in time between kinetic and internal energy. Ignoring viscosity, those waves are undamped and entirely time-reversible.

<u>Problem 2.</u> (a) We see immediately that

$$J^{(1)}_{i0} = -L_{i0,j\nu}\, \partial_j \lambda_\nu = 0$$

since $L_{i0,j\nu} = 0$ for all $i,j = 1,2,3$ and $\nu = 0,1,2,3,4$.

(b) From the definitions $\underset{i=1,2,3}{\lambda_i = \frac{u_i}{T}}$ and $\lambda_4 = -\frac{1}{T}$, we see that

$$\partial_j \lambda_\ell = -\frac{1}{T^2}\partial_j T\, u_\ell + \frac{1}{T}\partial_j u_\ell, \qquad \partial_j \lambda_4 = \frac{1}{T^2}\partial_j T$$

Thus,

$$J^{(1)}_{ik} = L_{ik,j\beta}\,\partial_j \lambda_\beta = L_{ik,j\ell}\,\partial_j \lambda_\ell + L_{ik,j4}\,\partial_j \lambda_4$$

$$= \left[\eta T \left(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj} - \frac{2}{3}\delta_{ik}\delta_{j\ell} \right) + 5T\delta_{ik}\delta_{j\ell} \right]$$

$$\times \left[-\frac{1}{T^2}\partial_j T\, u_\ell + \frac{1}{T}\partial_j u_\ell \right]$$

$$+ \left[\eta T \left(u_i \delta_{kj} + u_k \delta_{ij} - \frac{2}{3}\delta_{ik} u_j \right) \right] \cdot \left(\frac{1}{T^2}\partial_j T \right)$$

$$= -\left[\frac{\eta}{T}\left(u_i \partial_k T + u_k \partial_i T - \frac{2}{3}\delta_{ik}(u\cdot\nabla)T \right) + \frac{5}{T}\delta_{ik}(u\cdot\nabla)T \right]$$

$$+ \left[\eta \left(\partial_i u_k + \partial_k u_i - \frac{2}{3}(\nabla\cdot u)\delta_{ik} \right) + 5\,\delta_{ik}(\nabla\cdot u) \right]$$

$$+ \left[\frac{\eta}{T}\left(u_i \partial_k T + u_k \partial_i T - \frac{2}{3}\delta_{ik}(u\cdot\nabla)T \right) + \frac{5}{T}\delta_{ik}(u\cdot\nabla)T \right]$$

$$= T^{(1)}_{ik} \quad \checkmark$$

(c) Next, in the same manner,

$$J_{i4}^{(1)} = L_{i4,j\beta}\,\partial_j\lambda_\beta = L_{i4,j\ell}\,\partial_j\lambda_\ell + L_{i4,j4}\,\partial_j\lambda_4$$

$$= \left[\eta T\left(u_j\delta_{\ell i} + u_\ell\delta_{ij} - \frac{2}{3}\delta_{j\ell}u_i\right) + \zeta T\,\delta_{j\ell}u_i\right] \times \left[-\frac{1}{T^2}(\partial_j T)u_\ell + \frac{1}{T}\cdot\partial_j u_\ell\right]$$

$$+ \left[(\kappa T + \eta u^2)T\,\delta_{ij} + \left(\zeta + \frac{1}{3}\eta\right)T u_i u_j\right]\cdot\frac{1}{T^2}\partial_j T$$

$$= \left[-\frac{\eta}{T}\left(u_i\cancel{(u\cdot\nabla)T} + \cancel{\partial_i T\cdot u^2} - \frac{2}{3}u_i\cancel{(u\cdot\nabla)T}\right) - \frac{\zeta}{T}\cancel{u_i(u\cdot\nabla)T}\right]$$

$$+ \left[\eta\left(u_j\partial_j u_i + u_j\partial_i u_j - \frac{2}{3}u_i(\nabla\cdot u)\right) + \zeta u_i(\nabla\cdot u)\right]$$

$$+ \left[\left(\kappa + \cancel{\frac{\eta}{T}u^2}\right)\partial_i T + \frac{1}{T}\cancel{\left(\zeta + \frac{1}{3}\eta\right)}u_i(u\cdot\nabla)T\right]$$

$$= u_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}(\nabla\cdot u)\right) + \zeta\,\delta_{ij}(\nabla\cdot u)\right] + \kappa\,\partial_i T$$

$$= u_j T_{ij}^{(1)} + s_i^{(1)} \qquad \text{with} \qquad s_i^{(1)} = \kappa\,\partial_i T$$

It is a consequence of this problem that the compressible Navier–Stokes equation can be written in the so-called <u>Onsager force–flux form</u>

$$\partial_t \rho_\mu + \partial_i \left(J^{(0)}_{i\mu}(\rho) - L_{i\mu, j\nu} \, \partial_j \lambda_\nu \right) = 0$$

where $J^{(0)}_{i\mu}(\rho)$ are the reversible Euler fluxes of the five conserved densities for $\mu = 0, 1, 2, 3, 4$

$$J^{(0)}_{i0} = \rho_0 u_i = \rho u_i$$

$$J^{(0)}_{ij} = p \, \delta_{ij} + \rho u_i u_j$$

$$J^{(0)}_{i4} = \left(h + \tfrac{1}{2} \rho u^2 \right) u_i$$

which can be shown to conserve entropy $S = \int s(\rho(x,t)) \, d^3x$:

$$0 = \int \partial_i \left(\frac{\partial s}{\partial \rho_\mu} \right)(x,t) \, J^{(0)}_{i\mu} = -\int \partial_i \lambda_\mu \, J^{(0)}_{i\mu}$$

whereas the dissipative fluxes produce entropy

$$\frac{dS}{dt} = \int d^3x \; L_{i\mu, j\nu}(\rho(x,t)) \, \partial_i \lambda_\mu(x,t) \, \partial_j \lambda_\nu(x,t)$$

$$\geq 0$$

Since the Onsager coefficients are the components of a symmetric, positive-definite matrix. This formulation of the macroscopic fluid equations permits the most compact and elegant formulation of the fluctuation-dissipation relation, the so-called Onsager Principle.

Onsager's principle was first proposed in his Nobel Prize winning paper on irreversible thermodynamics

L. Onsager, "Reciprocal relations in irreversible processes, II." Phys. Rev. 38 2265–2279 (1931)

and further developed with his student Machlup

L. Onsager & S. Machlup, "Fluctuations and irreversible processes, I." Phys. Rev. 91 1505–1512 (1953)

The essential idea is this: any chosen hydrodynamic time history $\{\rho(t) : -\infty < t < \infty\}$ may be obtained as a solution of the hydrodynamic equations if one adds an appropriate fluctuating current $J_{i\mu}^{F} = -L_{i\mu,j\nu}\,\partial_j U_\nu$ for an appropriate set of fluctuation potentials U_μ:

$$\partial_t \rho_\mu + \partial_i \left[J_{i\mu}^{(0)}(\rho) - L_{i\mu,j\nu}(\rho)\left(\partial_j \lambda_\nu + \partial_j U_\nu\right) \right] = 0$$

Then the probability to observe this time-history is related to the dissipation required to produce it:

$$\mathrm{Prob}\left(\{\rho(t) : -\infty < t < \infty\}\right) \propto \exp\left(-\frac{1}{4k_B} \int_{-\infty}^{+\infty} dt \int d^3x\, L_{i\mu,j\nu}(\rho)\, \partial_i U_\mu(x,t)\, \partial_j U_\nu(x,t) \right)$$

For more discussion, see G.L. Eyink, "Dissipation and large thermodynamic fluctuations," J. Stat. Phys. 61 533–572 (1990) and L. Bertini et al., "Minimum dissipation principle in stationary nonequilibrium states," J. Stat. Phys. 116 831–841 (2004) for an application to driven systems out of thermodynamic equilibrium.

Problem 3. (a) From the definition

$$G(x,\xi) = \int_0^{\xi} d\eta \left[F(x,\eta) - \overline{F}(x) \right]$$

we see that since $\max_{(x,\xi)} |F(x,\xi)| = M < +\infty$

$$|G(x,\xi)| \leq \int_0^{\xi} d\eta \left(|F(x,\eta)| + |\overline{F}(x)| \right)$$

$$\leq 2M \cdot \int_0^{\xi} d\eta = 2M\xi \leq 2M$$

since $\xi \in [0,1]$. We have used the fact that

$$|\overline{F}(x)| = \left| \int_0^1 F(x,\xi)\, d\xi \right| \leq M \cdot \int_0^1 dx = M .$$

We thus conclude that $|G(x,\xi)| \leq 2M$ for all (x,ξ), so that G is bounded.

We next note that $G(x,0) = 0$ by definition and that also

$$G(x,1) = \int_0^1 d\eta \left[F(x,\eta) - \overline{F}(x) \right]$$

$$= \underbrace{\int_0^1 d\eta\, F(x,\eta)}_{\overline{F}(x)} - 1 \cdot \overline{F}(x) = 0 .$$

Since $G(x,0) = G(x,1)$ for all x, we see that G is also periodic with period 1 in the variable ξ.

Finally, all of the x-dependence of G is through the x-dependence of $F(x,\xi)$ and $\bar{F}(x) = \int_0^1 d\xi\, F(x,\xi)$, which are both assumed to be smooth in x. Thus,

$$\partial_x^n G(x,\xi) = \int_0^\xi d\eta \left[\partial_x^n F(x,\xi) - \partial_x^n \bar{F}(x) \right]$$

gives the nth x-derivative of G for all n such that the nth x-derivative of $F(x,\xi)$ exists as a bounded function.

(b) Since

$$G(x,\tfrac{x}{\epsilon}) = \int_0^{x/\epsilon} \left[F(x,\eta) - \bar{F}(x) \right] d\eta ,$$

we see that

$$\frac{d}{dx} G(x,\tfrac{x}{\epsilon}) = \frac{1}{\epsilon}\left[F(x,\tfrac{x}{\epsilon}) - \bar{F}(x) \right] + \underbrace{\int_0^{x/\epsilon} \left[F_x(x,\eta) - \bar{F}_x(x) \right] d\eta}_{G_x(x,\frac{x}{\epsilon})} .$$

Thus, the fundamental theorem of calculus implies that

$$G(b,\tfrac{b}{\epsilon}) - G(a,\tfrac{a}{\epsilon}) = \int_a^b dx\, \frac{d}{dx} G(x,\tfrac{x}{\epsilon})$$

$$= \frac{1}{\epsilon}\int_a^b \left[F(x,\tfrac{x}{\epsilon}) - \bar{F}(x) \right] dx + \int_a^b G_x(x,\tfrac{x}{\epsilon}) dx .$$

Solving for the F-integral we find that

$$\int_a^b \left[F(x, \tfrac{x}{\epsilon}) - \overline{F}(x) \right] dx =$$

$$\epsilon \left[G(b, b/\epsilon) - G(a, \tfrac{a}{\epsilon}) - \int_a^b dx\, G_x(x, \tfrac{x}{\epsilon}) \right].$$

Since G is a smooth function with all derivatives bounded, e.g.

$$| \partial_x^n G(x, \xi) | \leq B_n ,$$

we see that

$$\left| \int_a^b \left[F(x, \tfrac{x}{\epsilon}) - \overline{F}(x) \right] dx \right| \leq \epsilon \left[2B_0 + (b-a) B_1 \right]$$

$$\longrightarrow 0 \quad \text{as} \quad \epsilon \to 0.$$

Thus, we conclude that

$$\lim_{\epsilon \to 0} \int_a^b \left[F(x, \tfrac{x}{\epsilon}) - \overline{F}(x) \right] dx = 0. \qquad \text{QED.}$$

(c) We apply the previous result with $[a, b] = [x - \ell, x + \ell]$ to conclude that

$$\lim_{\epsilon \to 0} \left| \int_{x-\ell}^{x+\ell} F(y, \tfrac{y}{\epsilon}) dy - \int_{x-\ell}^{x+\ell} \overline{F}(y) dy \right| = 0$$

On the other hand, if we define

$$L_\nabla = \min_x \left| \frac{\overline{F}(x)}{\overline{F}_x(x)} \right|,$$

then for $\ell \ll L_\nabla$,

$$\int_{x-\ell}^{x+\ell} \overline{F}(y)\,dy \doteq 2\ell \cdot \overline{F}(x),$$

or, more precisely, we may choose ℓ_δ small enough that

$$\frac{1}{2\ell} \int_{x-\ell}^{x+\ell} \overline{F}(y)\,dy = \overline{F}(x) + O(\delta)$$

for $\ell < \ell_\delta$. For any such fixed ℓ, we can then also choose $\epsilon_\delta > 0$ such that for $\epsilon < \epsilon_\delta$

$$\begin{aligned} \int_{x-\ell}^{x+\ell} F(y, \tfrac{y}{\epsilon})\,dy &= \int_{x-\ell}^{x+\ell} \overline{F}(y)\,\delta y + O(2\ell \cdot \delta) \\ &= 2\ell \cdot \overline{F}(x) + O(2\ell \cdot \delta) \end{aligned}$$

so that $$\frac{1}{2\ell} \int_{x-\ell}^{x+\ell} F(y, \tfrac{y}{\epsilon})\,dy = \overline{F}(x) + O(\delta).$$

In other words, the coarse-grained field $\frac{1}{2\ell} \int_{x-\ell}^{x+\ell} F(y, \frac{y}{\epsilon})\,dy$ approximates the mean-field $\overline{F}(x)$ to any desired degree, for $\epsilon \ll \ell \ll L_\nabla$: