Q1.
You are interested in creating the following gross payoff profile using an option portfolio:

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff</td>
<td>10</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>30</td>
<td>10</td>
<td>0</td>
<td>-10</td>
</tr>
</tbody>
</table>

Consider only use call options, at what strikes, would you hold in your portfolio? There is more than One Way to solve this problem.

Q2.
You observe the following European option prices on a non-dividend paying stock. Suppose you know the term structure of interest rate is flat. Can you price a forward contract on the same stock for two years?

<table>
<thead>
<tr>
<th>$T$ (years)</th>
<th>Call</th>
<th>Puts</th>
<th>Strike</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>7</td>
<td>5</td>
<td>100</td>
</tr>
<tr>
<td>1.0</td>
<td>19</td>
<td>12</td>
<td>100</td>
</tr>
</tbody>
</table>

Q3.
ABC stock is currently trading at 100. In the next period, the price will either go up by 10% or down by 10%. The risk-free rate of interest over the period is 5%.
(a) Construct a replicating portfolio to value a call option written today with a strike price of 100. What is the hedge ratio? And use it to price the option.
(b) Calculate the risk-neutral probabilities in the model. Value the same call option using the risk-neutral probabilities. Check your answer with part (a).
(c) Using the risk-neutral probabilities, find the value of a put option written today, lasting one period, and with an exercise price of 100.
(d) Verify that the same price for the put results from put-call parity.

Q4.
You are given the following tree of stock prices. In addition, the rate of interest per period is constant at 2%. Price an American call and an American put. Both options are assumed to be at strike $45.$
Some Notes:
Why never exercise an American Call (on non-dividend stock) early while we can exercise an American Put early?
Suppose strike price is 50, if the stock goes up to 100, we do not exercise the American call because it might go higher when we exercise it. If the stock price goes to 0.1, we will exercise the American put because it is not likely to go down even further.
Solution 1.

<table>
<thead>
<tr>
<th>Strike</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity</td>
<td>1</td>
<td>1</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Solution 2.

We can set up the following two equations using the put-call parity condition:

\[ C - P = S - PV(K) \]

\[ 7 - 5 = S - 100 \cdot \exp(-0.5r) \quad 19 - 12 = S - 100 \cdot \exp(-1.0r) \]

We may solve these two equations for the stock price and interest rate.

The solutions are

\[ S = 96.72, \quad r = 0.1085. \]

Now we can easily price the forward using the cost-of-carry model, i.e.,

\[ F = S \cdot \exp(rT) = 96.72 \cdot \exp(0.1085 \cdot 2) = 120.17. \]

This is an example of how we can extract information about the equity and bond markets from the option markets.

Solution 3.

(a) Since \( K = 100 \), the payoffs from the call at maturity are:

\[ C_u = \max\{uS - K, 0\} = 10 \quad \text{and} \quad C_d = \max\{dS - K, 0\} = 0. \]

Let the replicating portfolio consist of \( \Delta \) units of the stock and borrowing/investment of $B. The initial cost of the portfolio is

\[ \Delta S + B = 100\Delta + B. \]

If the portfolio is to be replicate the call, we must have

\[ \Delta (uS) + \exp(r)B = C_u, \quad \Delta (dS) + rB = C_d \]

Substituting for these values, we obtain

\[ 110\Delta + \exp(0.05)B = 10, \quad 90\Delta + \exp(0.05)B = 0 \]

This implies \( \Delta = 1/2 \) and \( B = -42.8 \), i.e., the replicating portfolio involves a long position in a half unit of the stock and a borrowing of 42.8. Thus, the initial cost of the replicating portfolio is \( (1/2)100 - 42.8 = 7.2. \) Since the portfolio replicates the call, the call must also have an initial cost of 7.2.

(b) The risk-neutral probability \( p \) in this model is obtained by solving

\[ p(1.10) + (1 - p)(0.90) = \exp(0.05), \]

so \( p = 0.756. \) For this value of \( p \), the expected value of the call payoffs after one period are

\[ 0.756 \cdot (10) + 0.244 \cdot (0) = 7.56. \]

Discounting back to today, we obtain

\[ 7.56/\exp(0.05) = 7.2. \] Same with (a).

(c) The put pays \( P_u = 0 \) if the stock price moves up, and \( P_d = 10 \) if the stock price moves down. Using \( p \), the value of the put is, therefore,

\[ \exp(-r)(p \cdot P_u + (1 - p) \cdot P_d) = \exp(-0.05)(2.44) = 2.32. \]
(d) Put–call parity states that \( C + P V(K) = P + S \). Applied to the numbers here, this means 
\[ 7.2 + \exp(-0.05) \times 100 = P + 100, \text{ or } P = 7.2 + 95.12 - 100 = 2.32. \]

Solution 4.
We begin with the risk neutral probability. All q stands for the probability of a upward movement from the node.
Node at 45: \( q = \frac{\exp(0.02) - 2/3}{4/3 - 2/3} = 0.530 \)
Node at 60: \( qu = \frac{\exp(0.02) - 5/6}{8/6 - 5/6} = 0.374 \)
Node at 30: \( qd = \frac{\exp(0.02) - 2/3}{5/3 - 2/3} = 0.354 \)

For the call option. At maturity the payoffs will be \( \{35, 5, 0\} \) corresponding to stock prices \( \{80, 50, 20\} \).
The upper node at time 1 will have a call option value of 
\[ [35qu + 5(1 − qu)]\exp(-0.02) = 15.89. \] The lower node at time 1 will have a call option value of 
\[ [5qd + 0(1 − qd)] \exp(-0.02) = 1.733. \]
Using these two values we get the value of the call at time zero:
\[ [15.89q + 1.733(1 − q)] \exp(-0.02) = 9.057. \] Note that we do not need to check for early exercise since the stock does not pay dividends.
We now proceed onto the put option. At maturity the payoffs are \( \{0, 0, 25\} \) corresponding to stock prices \( \{80, 50, 20\} \).
If unexercised at that node, the value of the put at the upper node is given by 
\[ [0qu + 0(1 − qu)] \exp(-0.02) = 0. \] Since the exercise value is negative, the value of the American put at the upper node is the value of leaving it unexercised which is zero. If unexercised at the lower node, the value of the put at that node is 
\[ [0qd + 25(1 − qd)] \exp(-0.02) = 15.84. \] Exercising the put early results in a value of \( 45 − 30 = 15 \), which is less than the value of the put if unexercised. Hence, the value of the American put at this node is 15.84. Using these two values we get the value of the put at time zero:
\[ [0q + 15.84(1 − q)] \exp(-0.02) = 7.29. \]