Multivariate Records

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Overview

(I) Univariate and Multivariate Records
(II) Conditional Sampling of Multivariate Records
(II) Some Intriguing Phenomena
(IV) Record Waiting Times via Abstract Tubes
(V) Conclusions
Key Points

- Study of multivariate records is interesting
- It is challenging to obtain analytical results
- Numerical experiments can be designed
- Sampling from a union - importance sampling
- Computing probability of unions - abstract tubes
- Interesting conjectures resulting from experimentation
Univariate and Multivariate Records

Definition. Given \( x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, \ldots, \in \mathbb{R} \), we say a record occurs at index \( i \) if \( x^{(i)} \geq x^{(j)} \) for all \( j < i \).

Notation. For \( x, y \in \mathbb{R}^d \) say \( x < y \) if \( x_i < y_i \), for \( i = 1, \ldots, d \).

Definition. Given \( x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, \ldots, \) in \( \mathbb{R}^d \), we say that a multivariate record occurs at index \( i \) provided \( x^{(i)} \not< x^{(j)} \) for all \( j < i \).

Equivalently. \( x^{(i)} \) is Pareto optimal among \( x^{(j)}, j = 1, \ldots, i \).
Another Multivariate Record Notion

Alternative definition. Given \(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, \ldots,\) in \(\mathbb{R}^d\), we might say that a multivariate record occurs at index \(i\) provided \(x^{(j)} < x^{(i)}\) coordinatewise for all \(j < i\).

Observation. If \(X^{(i)}, i = 1, 2, 3, \ldots\) are iid, with \(X^{(i)}\) having iid components, the probability of a record at time \(n\) is

\[
P[X^{(1)}_i, X^{(2)}_i, \ldots, X^{(n-1)}_i < X^{(n)}_i, i = 1, \ldots, d] = (1/n)^d
\]

a summable series for \(d \geq 2\). By the Borel-Cantelli lemma at most finitely many such records with probability 1.
Men & Women’s Long Jump Records
1978-2008

- Men: 8.50
- Women: 7.75

Plot showing a single data point for Men at 8.50 and Women at 7.75.
Men & Women’s Long Jump Records
1978-2008

Men

Women

6.75
7.00
7.25
7.50
7.75

8.00
8.25
8.50
8.75

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Men & Women’s Long Jump Records
1978-2008
Men & Women’s Long Jump Records
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1978-2008
Men & Women’s Long Jump Records
1978-2008

[Graph showing comparison of men's and women's long jump records from 1978 to 2008]
Men & Women’s Long Jump Records
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1978-2008
Univariate Records: A Brief Survey
(See Arnold et al. Records)

**Classical model:** \( X_1, X_2, \ldots, \text{iid continuous } F \text{ in } (0, +\infty) \),

**Associated processes:**
- \( T_n = \text{index of } n\text{-th record}, \ n = 0, 1, 2, \ldots \)
- \( R_n = X^*_{T_n} = \text{record value sequence} \)
- \( J_0 = R_0, \ J_n = R_n - R_{n-1}, \ n \geq 1 \) increment process
- \( \Delta_n = T_n - T_{n-1} \) the inter-record time sequence
- \( N_n = \text{number of records among } \{X_1, X_2, \ldots, X_n\} \)

**Invariance:** Distributions of \( T_n, \Delta_n, N_n \) don’t depend on \( F \)
The Exponential Case

Transformation: $X_i^* := -\log(1 - F(X_i))$ are iid $Exp(1)$ rv's

Consequence of memoryless property:
Increments $R_0^*, R_1^* - R_0^*, R_2^* - R_1^*, \ldots$ are iid $Exp(1)$

- leads to tractable joint distribution for the records
- $R_n^* \sim \Gamma(n + 1, 1), \ i = 0, 1, 2, \ldots$
- leads to expression for distribution of $n$-th record in the non-exponential case, amenable to asymptotic analysis
Record Times

Define \( I_1 = 1 \) and for \( n > 1 \) let \( I_n = \begin{cases} 1 & \text{if } X_n \text{ is a record} \\ 0 & \text{otherwise} \end{cases} \)

- \( P[I_1 = 1, \ldots, I_n = 1] = P[X_1 < X_2 < \ldots < X_n] = 1/n! \)
- \( P[I_i = 1] = P[X_i = \max\{X_1, \ldots, X_i\}] = 1/i, \)
- \( P[I_1 = 1, \ldots, I_n = 1] = \prod_{i=1}^{n} P[I_i = 1] \)

Consequence: \( I_j \) are independent with \( I_j \sim \text{Bernoulli}(1/j) \)
Some basic facts:

- $N_n = \sum_{j=1}^{n} I_j = \text{number of records among } X_1, \ldots, X_n$
- $E(N_n) = \sum_{j=1}^{n} \frac{1}{j} = \log n + \gamma + o(1)$
- $Var(N_n) = \sum_{j=1}^{n} \frac{1}{j} \left(1 - \frac{1}{j}\right) = \log n + \gamma - \pi^2/6 + o(1)$
- $N_n \sim \log n \text{ w.p. } 1$
- $\frac{N_n - \log n}{\sqrt{\log n}} \xrightarrow{d} N(0, 1)$
- $(\log T_n)/n \xrightarrow{p} 1$

Records become somewhat rare
Remaining Records

Univariate case: A record at time $n$ always beats the best record up to that point in time. Consequently, the new record is always the only remaining record.

Multivariate case: A record at time $n$ may or may not beat some existing records. Consequently, there can be multiple remaining records.

Questions: How many records do we expect to have seen by time $n$? ($R_n$)

Questions: How many remaining records do we expect there to be at time $n$? (Call this $r_n$.)
Number of Remaining Bivariate Records
Number of Remaining Bivariate Records
Remaining bivariate records correspond to \( Y \) records in the usual sense after \( X \) reverse ordering.
Concomitants

**Definition:** Given iid bivariate rv’s \((X_i, Y_i), i = 1, \ldots, n\) from a continuous distribution in \(\mathbb{R}^{+2}\), define a permutation \(\pi\) by

\[
X_{\pi_1} < X_{\pi_2} < \cdots < X_{\pi_n}
\]

then \(Y_{[i:n]} = Y_{\pi_i}\) is referred to as the *concomitant* associated with the order statistic \(X_{\pi_i}\).

**Observation:** Assuming \(X_i\) and \(Y_i\) are independent, the rv’s \(Y_{[n:n]}, Y_{[n-1:n]}, \ldots, Y_{[1:n]}\) form an iid sequence.

**Consequence:** The number of remaining bivariate records satisfies \(r_n^{(2)} = R_n^{(1)} \sim \log n\).
Generalization to Multivariate Case

The number of remaining multivariate records for a $d$-variate sequence $X^{(1)}, \ldots, X^{(n)}$ is the same as the number of $d - 1$-variate records for the sequence obtained by

- sorting in reverse order on any one of the coordinates,
- eliminating the coordinate used for sorting, and
- counting the number of remaining records for the resulting sequence.
Generalization to Multivariate Case

$$
\begin{bmatrix}
X_1^{(1)} & X_2^{(1)} & \cdots & X_d^{(1)} \\
X_1^{(2)} & X_2^{(2)} & \cdots & X_d^{(2)} \\
\vdots & \vdots & \cdots & \vdots \\
X_1^{(n-1)} & X_2^{(n-1)} & \cdots & X_d^{(n-1)} \\
X_1^{(n)} & X_2^{(n)} & \cdots & X_d^{(n-1)}
\end{bmatrix}
$$
Generalization to Multivariate Case

$$
\begin{bmatrix}
X_{1}^{(\pi_1)} & X_{2}^{(\pi_1)} & \ldots & X_{d}^{(\pi_1)} \\
X_{1}^{(\pi_2)} & X_{2}^{(\pi_2)} & \ldots & X_{d}^{(\pi_2)} \\
\vdots & \vdots & \ddots & \vdots \\
X_{1}^{(\pi_{n-1})} & X_{2}^{(\pi_{n-1})} & \ldots & X_{d}^{(\pi_{n-1})} \\
X_{1}^{(\pi_n)} & X_{2}^{(\pi_n)} & \ldots & X_{d}^{(\pi_n)}
\end{bmatrix}
$$
Generalization to Multivariate Case

\[
\begin{bmatrix}
X_1^{(\pi_1)} & X_2^{(\pi_1)} & \ldots & X_d^{(\pi_1)} \\
X_1^{(\pi_2)} & X_2^{(\pi_2)} & \ldots & X_d^{(\pi_2)} \\
\vdots & \vdots & \ddots & \vdots \\
X_1^{(\pi_{n-1})} & X_2^{(\pi_{n-1})} & \ldots & X_d^{(\pi_{n-1})} \\
X_1^{(\pi_n)} & X_2^{(\pi_n)} & \ldots & X_d^{(\pi_n)}
\end{bmatrix}
\]
Generating the Next Bivariate Record
(putting aside time needed to obtain it)

Assumption: \((X_i, Y_i)\) a pair of independent \(Exp(1)\) rv’s

\[
f_{XY}(x, y) = e^{-(x+y)}I_{(0, +\infty) \times (0, +\infty)}(x, y)
\]

Goal: Sample from \(f\) conditional on falling in the set of new record-breaking pairs:
Generating the Next Bivariate Record
(putting aside time needed to obtain it)

Assumption: \((X_i, Y_i)\) a pair of independent \(Exp(1)\) rv’s

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Orthant Decomposition and Conditional Sampling
Orthant Decomposition and Conditional Sampling
Orthant Decomposition and Conditional Sampling

The diagram illustrates the decomposition of the region $A(x_1, y_2)$ into record-breakers and non-record-breakers. The region $A(x_1, y_2)$ is divided into two parts: a green shaded area labeled "record-breakers" and a red shaded area labeled "non-record-breakers."
Orthant Decomposition and Conditional Sampling

\[ A(x_2, y_3) \]

- Record-breakers
- Non record-breakers
Orthant Decomposition and Conditional Sampling

- Record-breakers
- Non record-breakers

$A(x_3, y_4)$
Orthant Decomposition and Conditional Sampling

A(x₄, 0)

non record-breakers

record-breakers

x

y
Orthant Decomposition and Conditional Sampling

Orthant: \( A(u, v) = \{(x, y) : x > u, y > v\} \)

Orthant probability: \( P_f[A(u, v)] = (1 - e^{-u})(1 - e^{-v}) \)

Conditional sampling: To get \((X, Y) \mid (X, Y) \in A(u, v)\) we take \((X, Y) = (u, v) + (W, Z)\), where \(W, Z\) are independent \(Exp(1)\) rv's
Conditional Sampling

Acceptance/Rejection

**Goal:** Sample from given probability density $f$

**Idea:** Find probability density $g$ with:

(a) $g$ easy to sample from

(b) $f \leq cg$ for $c$ small
Conditional Sampling

Acceptance/Rejection

**Goal:** Sample from given probability density $f$

**Idea:** Find probability density $g$ with:

(a) $g$ easy to sample from

(b) $f \leq cg$ for $c$ small ($c = \text{expected # of iterations}$)
Conditional Sampling
Acceptance/Rejection

**Goal:** Sample from given probability density \( f \)

**Idea:** Find probability density \( g \) with:

(a) \( g \) easy to sample from

(b) \( f \leq cg \) for \( c \) small \((c = \text{expected } \# \text{ of iterations})\)

**Algorithm:**

Set Accept=False
Repeat
    Sample \( X \sim g \)
    Set Accept=True with prob \( f(X)/cg(X) \)
Until Accept=True
Return \( X \)
Illustration:
Sampling the Standard Normal Distribution

\[ f(x) \quad \text{and} \quad cg(x) \]
Illustration:
Sampling the Standard Normal Distribution
Illustration:
Sampling the Standard Normal Distribution
Illustration: Sampling the Standard Normal Distribution

The diagram illustrates a standard normal distribution with the probability density function (PDF) labeled as $f(x)$ and a modified distribution labeled as $cg(x)$. The distribution is centered at $x=0$, with values ranging from $-3$ to $3$ on the x-axis.
Application to Union Sampling

Want to sample $X$ according to pdf $f$, conditioned on $X \in \bigcup_{i=1}^{n} A_i$

$$\tilde{f} := \frac{f \mathbb{1}_{\bigcup_{i=1}^{n} A_i}}{P_f(\bigcup_{i=1}^{n} A_i)}$$

where

$$P_f(B) := \int_B f(x) dx.$$
Application to Union Sampling

Want to sample $X$ according to pdf $f$, conditioned on $X \in \bigcup_{i=1}^{n} A_i$

$$\tilde{f} := \frac{f I_{\bigcup_{i=1}^{n} A_i}}{P_f(\bigcup_{i=1}^{n} A_i)}$$

where

$$P_f(B) := \int_{B} f(x) dx.$$
Union Sampling

\[\tilde{f} = \frac{f(x)I_{\bigcup_{i=1}^{n} A_i(x)}}{P_f(\bigcup_{i=1}^{n} A_i)} = \frac{\sum_j P_f(A_j)}{P_f(\bigcup_{i=1}^{n} A_i)} \frac{1}{\sum_j I_{A_j}(x)} \sum_i \frac{P_f(A_i)}{\sum_j P_f(A_j)} \frac{f I_{A_i}}{P_f(A_i)}\]

\[= \frac{K}{N(x)} \sum_{i=1}^{n} q_i \tilde{f}_i(x)\]
Union Sampling

\[ \tilde{f} = \frac{f(x)I_{\bigcup_{i=1}^{n} A_i(x)}}{P_f(\bigcup_{i=1}^{n} A_i)} = \frac{\sum_{j} P_f(A_j)}{P_f(\bigcup_{i=1}^{n} A_i)} \sum_{i} \frac{1}{I_{A_j}(x)} \sum_{i} P_f(A_i) \frac{fI_{A_i}}{P_f(A_i)} \]

\[ = \frac{K}{N(x)} \sum_{i=1}^{n} q_i \tilde{f}_i(x) \]

\[ K := \frac{\sum_{j=1}^{n} P_f(A_j)}{P_f(\bigcup_{j=1}^{n} A_j)} \]

\[ q_i = \frac{P_f(A_i)}{\sum_{j=1}^{n} P_f(A_j)} \]

\[ N(x) = \sum_{j=1}^{n} I_{A_j}(x), \]

\[ \tilde{f}_i = \frac{I_{A_i}(x)f(x)}{P_f(A_i)} \]
Union Sampling

\[
\begin{align*}
\tilde{f} &= \frac{f(x)I_{\bigcup_{i=1}^{n} A_i}(x)}{P_f(\bigcup_{i=1}^{n} A_i)} = \frac{\sum_j P_f(A_j)}{P_f(\bigcup_{i=1}^{n} A_i)} \frac{1}{\sum_j I_{A_j}(x)} \sum_i \frac{P_f(A_i)}{\sum_j P_f(A_j)} \frac{fI_{A_i}}{P_f(A_i)} \\
&= \frac{K}{N(x)} \sum_{i=1}^{n} q_i \tilde{f}_i(x)
\end{align*}
\]

\[
K := \frac{\sum_{j=1}^{n} P_f(A_j)}{P_f(\bigcup_{j=1}^{n} A_j)} \quad N(x) = \sum_{j=1}^{n} I_{A_j}(x),
\]

\[
q_i = \frac{P_f(A_i)}{\sum_{j=1}^{n} P_f(A_j)} \quad \tilde{f}_i = \frac{I_{A_i}(x)f(x)}{P_f(A_i)}
\]
Union Sampling

\[ \tilde{f} = \frac{f(x) I_{\bigcup_{i=1}^{n} A_i}(x)}{P_f(\bigcup_{i=1}^{n} A_i)} = \frac{\sum_j P_f(A_j)}{P_f(\bigcup_{i=1}^{n} A_i)} \sum_j \frac{1}{I_{A_j}(x)} \sum_i \frac{P_f(A_i)}{\sum_j P_f(A_j)} \frac{f I_{A_i}}{P_f(A_i)} \]

\[ = \frac{K}{N(x)} \sum_{i=1}^{n} q_i \tilde{f}_i(x) \]

\[ K := \frac{\sum_{j=1}^{n} P_f(A_j)}{P_f(\bigcup_{j=1}^{n} A_j)} \]

\[ q_i = \frac{P_f(A_i)}{\sum_{j=1}^{n} P_f(A_j)} \]

\[ N(x) = \sum_{j=1}^{n} I_{A_j}(x), \]

\[ \tilde{f}_i = \frac{I_{A_i}(x)f(x)}{P_f(A_i)} \]
Union Sampling

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\tilde{f} = \frac{f(x)I_{\bigcup_{i=1}^{n} A_i}(x)}{P_f(\bigcup_{i=1}^{n} A_i)} = \frac{\sum_j P_f(A_j)}{P_f(\bigcup_{i=1}^{n} A_i)} \sum_j \frac{1}{I_{A_j}(x)} \sum_i \frac{P_f(A_i)}{\sum_j P_f(A_j)} \frac{f I_{A_i}}{P_f(A_i)}
\]

\[= \frac{K}{N(x)} \sum_{i=1}^{n} q_i \tilde{f}_i(x)\]

\[K := \frac{\sum_{j=1}^{n} P_f(A_j)}{P_f(\bigcup_{j=1}^{n} A_j)}\]

\[q_i = \frac{P_f(A_i)}{\sum_{j=1}^{n} P_f(A_j)}\]

\[N(x) = \sum_{j=1}^{n} I_{A_j}(x),\]

\[\tilde{f}_i = \frac{I_{A_i}(x)f(x)}{P_f(A_i)}\]
Union Sampling

\[
\tilde{f} = \frac{f(x)I_{\bigcup_{i=1}^{n} A_i(x)}}{P_f(\bigcup_{i=1}^{n} A_i)} = \frac{\sum_j P_f(A_j)}{P_f(\bigcup_{i=1}^{n} A_i)} \frac{1}{\sum_j I_{A_j}(x)} \sum_i \frac{P_f(A_i)}{\sum_j P_f(A_j)} \frac{fI_{A_i}}{P_f(A_i)}
\]

\[
= \frac{K}{N(x)} \sum_{i=1}^{n} q_i \tilde{f}_i(x)
\]

\[
K := \frac{\sum_{j=1}^{n} P_f(A_j)}{P_f(\bigcup_{j=1}^{n} A_j)} \quad N(x) = \sum_{j=1}^{n} I_{A_j}(x),
\]

\[
q_i = \frac{P_f(A_i)}{\sum_{j=1}^{n} P_f(A_j)} \quad \tilde{f}_i = \frac{I_{A_i}(x)f(x)}{P_f(A_i)}
\]
Algorithm

Write:

\[ \tilde{f} = \frac{K}{N(x)} \sum_{i=1}^{n} q_i \tilde{f}_i(x) \leq K \sum_{i=1}^{n} q_i \tilde{f}_i(x) \]

Algorithm: Repeatedly sample from the mixture distribution

\[ X \sim g = \sum_{i=1}^{n} q_i \tilde{f}_i(x), \]

and stop and keep \( X \) with probability

\[ \tilde{f}(X)/Kg(X) = 1/N(X) \]
Sampling from $g$

$$g = \sum_{i=1}^{n} q_i \tilde{f}_i(x)$$

This is a mixture of conditional densities. To sample from $g$

- pick an index $I$ s.t. $P[I = i] = q_i$
- sample $X$ from $\tilde{f}_I$
Some Interesting (Unexplained) Phenomena

Simulation experiment: Generate $N$ conditional new bivariate exponential records

Remark: $N$ equals $R_n$ for some unobserved (and no doubt huge) value of $n$
Remaining Bivariate Records After Sampling 10,000 Conditional Bivariate Exponential Records
**Behavior of Remaining Records:**  
**Bivariate Exponential Case**

Simulation experiment results:

<table>
<thead>
<tr>
<th>$R_n$</th>
<th>$r_n$</th>
<th>$x + y$</th>
<th>$\sqrt{2R_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5,000</td>
<td>109</td>
<td>100</td>
<td>100.0</td>
</tr>
<tr>
<td>10,000</td>
<td>133</td>
<td>142</td>
<td>141.4</td>
</tr>
<tr>
<td>50,000</td>
<td>338</td>
<td>316</td>
<td>316.2</td>
</tr>
<tr>
<td>100,000</td>
<td>428</td>
<td>447</td>
<td>447.2</td>
</tr>
<tr>
<td>200,000</td>
<td>592</td>
<td>632</td>
<td>632.5</td>
</tr>
</tbody>
</table>

Apparent relationship: For $d = 2$

\[ r_n \sim \sqrt{2R_n} \quad R_n \sim \frac{1}{2}r_n^2 \]
Behavior of Remaining Records:
Trivariate Exponential Case

Simulation experiment results:

<table>
<thead>
<tr>
<th>$R_n$</th>
<th>$r_n$</th>
<th>$\frac{1}{2}(6R_n)^{\frac{2}{3}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>19</td>
<td>17.1</td>
</tr>
<tr>
<td>40</td>
<td>34</td>
<td>28.7</td>
</tr>
<tr>
<td>60</td>
<td>34</td>
<td>39.0</td>
</tr>
<tr>
<td>80</td>
<td>56</td>
<td>48.3</td>
</tr>
</tbody>
</table>

Apparent relationship: For $d = 3$

$$r_n \sim \frac{1}{2}(3!R_n)^{\frac{2}{3}}$$
Conjecture in General Case

**Conjecture:**  \( R_n \sim \frac{1}{d!} (\log n)^d \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>( R_n )</th>
<th>( r_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \log n )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} (\log n)^2 )</td>
<td>( \log n )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{3!} (\log n)^3 )</td>
<td>( \frac{1}{2} (\log n)^2 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Distribution of Beaten Records

**Experiment:** Generate 10,000 conditional bivariate records & tally for each record, $B$ the number of remaining records it beat.

**Empirical distribution of $B$:**

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$\geq 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_m$</td>
<td>.492</td>
<td>.260</td>
<td>.126</td>
<td>.065</td>
<td>.032</td>
<td>.012</td>
<td>.0077</td>
<td>.003</td>
<td>.0017</td>
<td>.0016</td>
</tr>
</tbody>
</table>

Remarkably similar to a Geometric $(1/2)$ distribution
Sampling only *records* leads to useful information about relationship between $R_n$ and $r_n$.

Want to understand the relationship of these random variables to $n$.

Still want to avoid explicit and time-consuming sampling of non-records.

Sufficient to determine behavior of waiting time to next record.
Sampling Waiting Time to Next Record

Waiting time distribution: Given current remaining records, $T \sim Geometric(p)$ where $p = P[(X, Y) \in \mathcal{N}^c]$. 
Unions of Translated Orthants

Decompose: $\mathcal{N}^c = \bigcup_{i=1}^{n} A_i$

Inclusion-Exclusion Identity: $I_{\mathcal{N}^c} = \sum_{i=1}^{n} I_{A_i} - \sum_{i=1}^{n-1} I_{A_i \cap A_{i+1}}$

\[
P_f[\mathcal{N}^c] = \sum_{i=1}^{n} P_f[A_i] - \sum_{i=1}^{n-1} P_f[A_i \cap A_{i+1}]
\]
Unions of Translated Orthants

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Inclusion-Exclusion Identity: \( I_{\mathcal{N}^c} = \sum_{i=1}^{n} I_{A_i} - \sum_{i=1}^{n-1} I_{A_i \cap A_{i+1}} \)

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P_f[\mathcal{N}^c] = \sum_{i=1}^{n} P_f[A_i] - \sum_{i=1}^{n-1} P_f[A_i \cap A_{i+1}]\]
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Unions of Translated Orthants

Illustration of the inclusion-exclusion identity
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

Indicator of $A_1$
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

Indicator of $A_2$

\[ N \]
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

Indicator of $A_3$
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

Indicator of $A_4$
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

Indicator of $A_5$
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

\[ T_1 = \text{sum of indicators of the } A_i \]
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

Indicator of $A_1 \cap A_2$
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

Indicator of $A_2 \cap A_3$
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

\[ T_2 = \text{sum of indicators of the } A_i \cap A_{i+1} \]
Unions of Translated Orthants

Illustration of the inclusion-exclusion identity

$T_1 - T_2$
Higher-Dimensional Case

3-d animation
Union of Translates of Orthants
Inclusion-Exclusion Identity for Orthants

Define $\mathcal{F}$ to be the collection of index sets $J$ for which $O_{u^{(i)}}, i \in J$ meet on $\partial X$. For generic choices of $u^{(i)}, i = 1, \ldots, n$ these sets have at most $d$ elements.

Theorem (N. & Wynn 2001). The collection of sets $\{O_{u^{(i)}}, i = 1, \ldots, n\}$ together with the simplicial complex $\mathcal{F}$ forms an abstract tube.
Consequences of the Abstract Tube Property

Can write

\[ I_{\bigcup_{i=1}^{n} O_u(i)} = \sum_{F \in \mathcal{F}} (-1)^{|F|+1} I_{\bigcap_{i \in F} O_u(i)} \]

- Number of terms grows like a polynomial in \( n \) instead of exponentially.
- Computational effort to calculate \( \mathcal{F} \) is order \( n^{d-1} \).
- Truncation inequality property.
Conclusions

- Multivariate record process is interesting to study, and little is known
- Computational tools for their study are available
- Interesting phenomena worthy of further investigation are being uncovered