\[ E_\eta \psi(x) = \int \psi(x) p(x|\eta) \, dx \]

\[ \frac{\partial}{\partial \eta} E_\eta \psi(x) = \int \psi(x) \frac{\partial}{\partial \eta} p(x|\eta) \, dx = \int \psi(x) \frac{\partial}{\partial \eta} \exp \left( \eta x - A(\eta) \right) \, dx \cdot \frac{\partial}{\partial \eta} \left( \eta x - A(\eta) \right) \]

\[ = \int \psi(x) (x - \dot{A}(\eta)) p(x|\eta) \, dx = E_\eta \left[ \psi(x) (x - \dot{A}(\eta)) \right] \]

\[ = E_\eta \left[ \psi(x) (x - E_\eta(x)) \right] = \text{Cov}_\eta (\psi(x), x). \]

Now take \( x' \) independent \( \sim p(x|\eta) \).

Then since \( \psi \) is non-decreasing we have

\[ x' \leq x \Rightarrow \psi(x') = \psi(x) \]
\[ x \leq x' \Rightarrow \psi(x) = \psi(x') \]

which implies

\[ (x' - x) (\psi(x') - \psi(x)) \geq 0 \quad \text{always.} \]

Taking expectations

\[ 0 \leq E_\eta [(x' - x) (\psi(x') - \psi(x))] \]

\[ = E_\eta \left[ (x' - \mu) - (x - \mu) \right] (\psi(x') - \psi(x)) \quad \text{when } \mu = E x \]

\[ = \text{Cov}_\eta (x, \psi(x)) + \text{Cov}_\eta (x, \psi(x)) - \text{Cov}_\eta (x, \psi(x)) - \text{Cov}_\eta (x, \psi(x)) = 2 \text{Cov}_\eta (x, \psi(x)) \]

by independence of \( x' \) \( \sim \)

\[ = 2 \frac{2}{\eta} E_\eta \psi(x) \quad \text{by the above.} \]