Problem 2. (1.1.13)

(a) Note that for \( T \sim F_0 \) with \( F_0' = f_0 \) the hazard function is by definition

\[
h_0(t) = \frac{f_0(t)}{1 - F_0(t)} = -\frac{1}{dt} \log(1 - F_0(t)) = -\frac{1}{dt} S_0(t)
\]

so we can write

\[
S_0(t) = C - \int_{s=0}^{t} h_0(s) \, ds 
\]

for some arbitrary \( C \in \mathbb{R} \)

hence

\[
S_0(t) = \exp \left\{ C - \int_{s=0}^{t} h_0(s) \, ds \right\}
\]

and since

\[
F_0(0) = 0, \quad S_0(0) = 1
\]

we have \( C = 0 \) and

\[
S_0(t) = \exp \left\{ -\int_{s=0}^{t} h_0(s) \, ds \right\}.
\]

Now if we assume

\[
h(\alpha | x) = h_0(x) \exp \{ g(\rho, x) \}
\]

then using the above we see that

\[
S(\alpha | x) = \exp \left\{ -\int_{s=0}^{t} h_0(x) \exp \{ g(\rho, x) \} \, ds \right\} = \exp \left\{ -\exp \left\{ g(x) \right\} \int_{s=0}^{t} h_0(s) \, ds \right\}
\]

\[
= \left( \exp \left\{ -\int_{s=0}^{t} h_0(s) \, ds \right\} \right)^{\exp \{ g(x) \}} = S_0(t)^{\Delta}
\]

(b) Define

\[
Q(t) = -\log S_0(t)
\]

which is strictly increasing since \( S_0 = 1 - F_0 \) is strictly decreasing.

Then

\[
P(\hat{Q}(\hat{y}) > t) = P(-\log \hat{S}_0(\hat{y}) > t) = P(\hat{S}_0(\hat{y}) > \hat{y}^t) = P(y > \hat{S}_0(\hat{y}^t))
\]

\[
= S(\hat{S}_0(\hat{y}^t)) = [S_0(\hat{S}_0(\hat{y}^t))]^{\Delta} = \hat{y}^{t \Delta}
\]

and this does not depend on \( h_0 \).

(c) If \( \hat{Q} \) is the transformation in (b) we have

\[
P(\hat{Q}(\hat{y}) > t) = \hat{y}^{t \Delta}
\]
so the cdf of $Q(Y)$ is given by

$$F_Q(y) = 1 - e^{t\Delta}$$ for $t > 0$.

The inverse cdf is therefore

$$-\frac{1}{\Delta} \log(1-u)$$

therefore we can write

$$Q(Y) = -\Delta \log(1-U) \quad \text{where} \quad U \sim \text{Uniform}(0,1).$$

Thus

$$-\log Q(Y) = \log(\Delta) - \log(-\log(1-U))$$

$$= g(p, z) - \log(-\log(1-U))$$

so we see that

$$-\log Q(Y) = g(p, z) + \epsilon$$

where

$$\epsilon = -\log(-\log(1-U))$$

This random variable $\epsilon$ has cdf

$$P[\epsilon \leq t] = P[-\log(-\log(1-U)) \leq t] = P[\log(1-U) \leq -e^{t}]$$

$$= P[1-U \leq e^{t}] = e^{t} \quad \text{for } t > 0$$

Note: $1-U \sim \text{Uniform}(0,1)$