To see that $\text{Var}(X) \& \text{Var}(Y)$ are positive definite, note since

$$u^\top \text{Var}(X)u = \text{Var}(u^\top X) \geq 0$$

for any $u$. So $\text{Var}(X)$ is positive semi-definite.

Observe that $(X, Y)$ has covariance matrix

$$\text{Var}(X, Y) = \begin{pmatrix} \text{Var}(X) & \Sigma(X,Y) \\ \Sigma(X,Y)^\top & \text{Var}(Y) \end{pmatrix}$$

and in addition

$$\text{Var}(u^\top X + v^\top Y) = (u^\top v)^\top \text{Var}(X) \begin{pmatrix} u \\ v \end{pmatrix} \geq 0$$

for all choices of $u \& v$.

Now if $u^\top \text{Var}(X)u = 0$ then for any $v$ we have

$$(u^\top \begin{pmatrix} \text{Var}(X) & I \\ I & \text{Var}(Y) \end{pmatrix} v) \geq 0$$

which gives

$$u^\top \text{Var}(X)u + 2u^\top v + v^\top \text{Var}(Y)v \geq 0$$

for all $v$. If we take $v = cu$ then we obtain

$$2c u^\top u + c^2 u^\top \text{Var}(Y)u \geq 0 \tag{\ast}$$

for all $c \in \mathbb{R}$. If $u \neq 0$ then $u^\top u > 0$. By taking $c \neq 0$ and negative, the expression on the left-hand side of $(\ast)$ is negative, which is a contradiction. We conclude that $u = 0$. Thus, we have shown

$$u^\top \text{Var}(X)u = 0 \implies u = 0$$
hence $\text{Var}(X)$ is positive definite. The same argument shows $\text{Var}(Y)$ is positive definite.

Finally, we show $M = \text{Var}(X) - \text{Var}(Y)'$ is positive semi-definite. $M$ is clearly a symmetric matrix since $\text{Var}(X)$ & $\text{Var}(Y)$ are.

Next, if we take $v = -\text{Var}(Y)^{-1}u$ we see that

$$0 \leq (u')^T \begin{pmatrix} \text{Var}(X) & \text{I} \\ \text{I} & \text{Var}(Y) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = u^T \text{Var}(X) u + 2u^T v + v^T \text{Var}(Y)v$$

$$= u^T \text{Var}(X) u - 2u^T \text{Var}(Y)^{-1}u +$$

$$+ (-u^T \text{Var}(Y)^{-1}) \text{Var}(Y)(\text{Var}(Y)^{-1}u)$$

$$= u^T \text{Var}(X) u - u^T \text{Var}(Y)^{-1}u$$

$$= u^T [\text{Var}(X) - \text{Var}(Y)^{-1}] u$$

$$= u^T Mu$$

\[\square\]