Dynamical Systems (550.391)
Take-Home Project II Solutions

1. **(40 pts).** A lake is initially stocked with 100 bass and 600 redear. There is ample food for the redear. Because bass prey on redear, the population of bass will increase at a rate, p, proportional to the number of encounters between the species; bass will also die at a rate, q, proportional to the bass population.

   The redear multiply at a rate, u, proportional to their population and die off at a rate, v, proportional to the number of encounters between the two species.

   Let $B(t)$ be the bass population at time t and let $R(t)$ be the redear population at time t.

   (a) Create a system of two first order ODEs to model the population dynamics.

   (b) Suppose it is know that $p = 0.00004$, $q = 0.02$, $u = 0.05$, and $v = 0.0004$. Find the fixed points for this system and determine their type and stability.

   (c) Using the theory from Chapter 7, determine if the system has any limit cycles, or if it is possible to rule out periodic solutions.

   (d) Use the Runge-Kutta method to plot $R(t)$ and $B(t)$ for $0 < t < t_f$. (You should choose t_f and the step size appropriately.)

   (e) Based on the plots in part (d) estimate the period of oscillation for each species.

   ANSWERS: (The following uses the correct value for v)

   (a) Create a system of two first order ODEs to model the population dynamics.

$$\begin{aligned} \frac{dB}{dt} &= -qB + pBR \\ \frac{dR}{dt} &= uR - vBR \end{aligned}$$

   (b) The equation for dB/dt says that $dB/dt = 0$ when $B = 0$ or $R = 500$. Plugging these values into the second equation yields two fixed points (B, R): (0,0) and (125,500).

   The Jacobian is given by

$$J(B,R) = \begin{bmatrix} -0.02 + 0.00004R & 0.00004B \\ -0.0004R & 0.05 - 0.0004B \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} -0.02 & 0 \\ 0 & 0.05 \end{bmatrix}$$

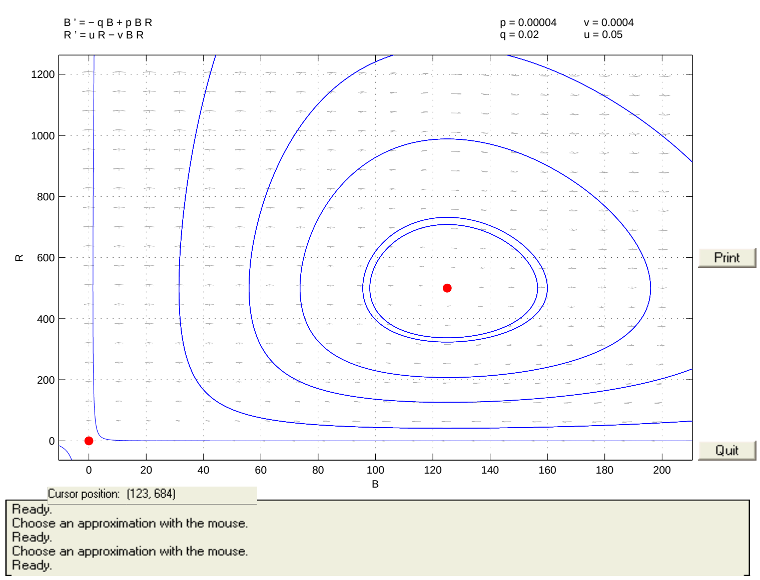
so the evalues are $\lambda_1 = -0.02$ and $\lambda_2 = 0.05$; the origin is an unstable saddle point.

$$J(125, 500) = \begin{bmatrix} 0 & 0.0005 \\ -0.2 & 0 \end{bmatrix}$$

so the evalues are $\lambda_1 = 0.01i$ and $\lambda_2 = -0.01i$; this is a borderline case. However, we can solve for the trajectories exactly:

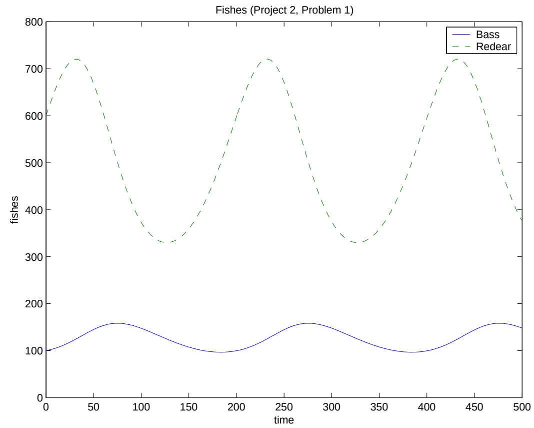
$$\begin{aligned} \frac{dB}{dR} &= \frac{B(-q+pR)}{R(u-vB)} \\ \frac{u-vB}{B}dB &= \frac{-q+pR}{R}dR \\ u\ln B - vB &= -q\ln R + pR + c \\ u\ln B - vB + q\ln R - pR &= c \end{aligned}$$

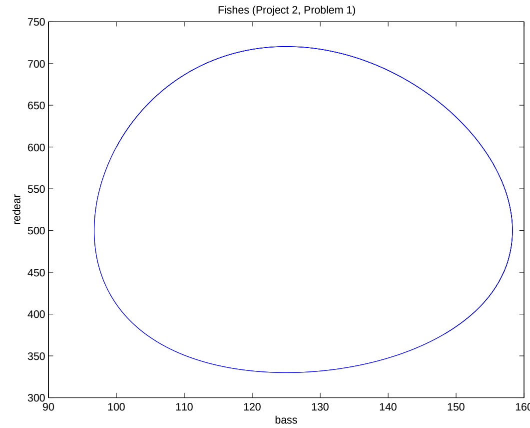
These are closed orbits, and the critical point is a center, as shown in the phase portrait below.



(c) One theorem says that a limit cycle must enclose a fixed point. Another theorem says that the fixed point enclosed by a limit cycle cannot be a saddle point. Hence, if there is a limit cycle, it must surround (125,500). But, we have shown this fixed point to be a center and a limit cycle is an *isolated* closed orbit. Hence, the system has no limit cycles.

(d) The Runge-Kutta code is available in a separate document on the website. Here are some of the graphics generated:





(e) The period of oscillation is about 200 time units for each population.

2. **(40 pts).** Consider the predator-prey model

$$\frac{dx}{dt} = x\left(1-\frac{x}{7}\right) - \frac{6xy}{7+7x}, \tag{1}$$

$$\frac{dy}{dt} = 0.2y\left(1-\frac{Ny}{x}\right), \tag{2}$$

where N is a constant. The functions $x(t) \neq 0$ and $y(t)$ represent the populations of prey and predators respectively.

(a) How does this model differ from the predator-prey models we have studied in the past?

(b) In the context of population modeling, interpret the terms

(i) $x(1-\frac{x}{7})$

(ii) $-\frac{6xy}{7+7x}$

(ii) $0.2y\left(1-\frac{Ny}{x}\right)$

(c) Analyze the population dynamics when $N = 2.5$. That is, determine the type and stability of any fixed points; determine the stability of any limit cycles; sketch or computer-generate a phase plane portrait.

(d) Use your analysis from part (c) to provide discuss the population dynamics for this system when $N = 2.5$.

(e) Analyze the population dynamics when $N = 0.5$.

(f) Use your analysis from part (e) to provide discuss the population dynamics for this system when $N = 0.5$.

(g) (Extra Credit!) Create a trapping region for the system when $N = 0.5$ and use the Poincare-Bendixson Theorem to prove the existence of a limit cycle.

ANSWERS:

(a) Many of the "constants of proportionality" are no longer constant. Instead they vary as a function of one of the populations being considered. (See part (b) for further details.)

(b) The term $x(1-\frac{x}{7})$ represents logistic growth for the prey population in the absence of predators. The term $-\frac{6xy}{7+7x}$ represents deaths of prey due to the interaction between the predator and prey populations. In this model, the "constant of proportionality" is $\frac{6}{7+7x}$. Thus, the effect of predators caps out at a maximum predation rate. Finally, the term $0.2y\left(1-\frac{Ny}{x}\right)$ denotes the predator growth rate, which would be logistic, except for the fact that the "constant of proportionality" is N/x. Thus, this term denotes the predator growth rate when a maximum of x/N predators is supported by x prey.

The Jacobian for this system is

$$J(x,y) = \begin{bmatrix} -2x/7 + 1 + 42xy/(7+7x)^2 - 6y/(7+7x) & -6x/(7+7x) \\ 0.2Ny/x^2 & 0.2 - 0.4Ny/x \end{bmatrix}$$

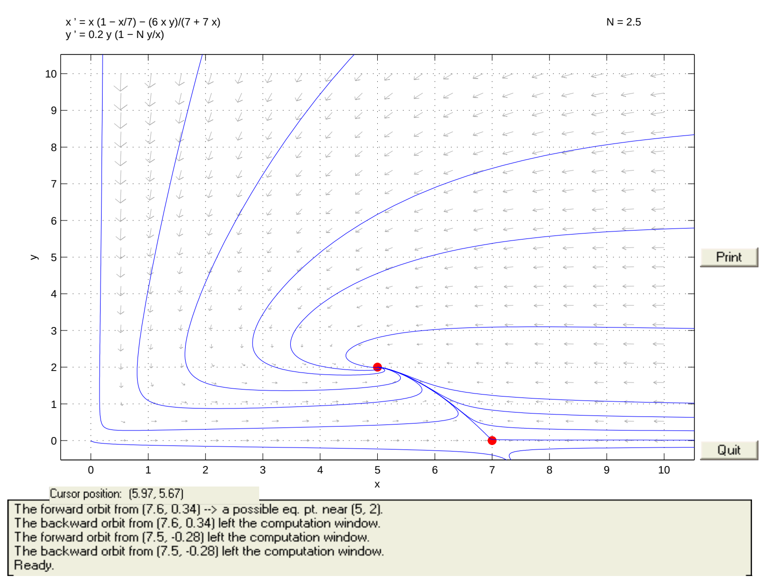
(c) For this value of N, there are two critical points in the first quadrant: (5,2) and (7,0).

$$J(5,2) = \begin{bmatrix} -0.4762 & -0.7143 \\ 0.08 & -0.20 \end{bmatrix}$$

The eigenvalues are $-0.3381 \pm 0.1951i$ so this is a stable spiral.

$$J(7,0) = \begin{bmatrix} -1 & -3/4 \\ 0 & 1/5 \end{bmatrix}$$

The eigenvalues are -1 and 1/5 so this is an unstable saddle. The phase portrait for the system when $N = 2.5$ is shown below.



(d) For most initial conditions, the populations eventually settle down to constant, nonzero values. If there are any natural disasters of diseases, the populations would both decrease but, assuming that neither becomes extinct, they would eventually return to the stable equilibrium values. There is no periodic behavior in this model.

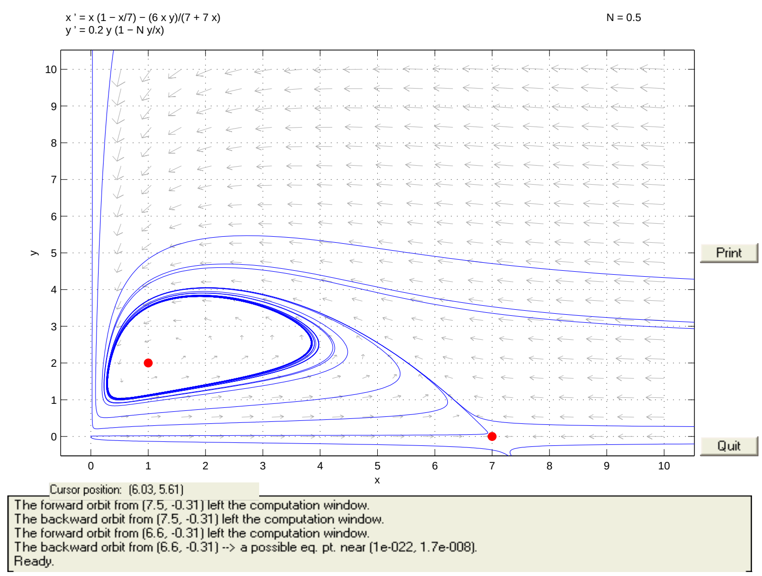
(e) For this value of N, there are two critical points in the first quadrant: (1,2) and (7,0).

$$J(1,2) = \begin{bmatrix} 0.2857 & -0.4286 \\ 0.4 & -0.20 \end{bmatrix}$$

The eigenvalues are $-0.0429 \pm 0.3353i$ so this is an unstable spiral.

$$J(7,0) = \begin{bmatrix} -1 & -3/4 \\ 0 & 1/5 \end{bmatrix}$$

The eigenvalues are -1 and 1/5 so this is an unstable saddle. The phase portrait for the system when $N = 0.5$ is shown below.



(f) All trajectories in the first quadrant are drawn to the limit cycle shown in the phase portrait. Therefore, no matter what the initial values of $x(t)$ and $y(t)$, the populations eventually rise and fall periodically. The limit cycle is stable.

3. **(40 pts).** The system

$$\frac{dx}{dt} = 3(x + y - \frac{x^3}{3} - k), \quad (3)$$

$$\frac{dy}{dt} = -\frac{1}{3}(x + 0.8y - 0.7), \quad (4)$$

is a special case of the *Fitzhugh-Nagumo* equations, which model the transmissions of neural impulses along an axon. The parameter k is known as the *external stimulus*.

(a) Show that system has only one critical point regardless of the value of k.

(b) Find the critical point for $k = 0$. Determine its type and stability. Use the computer to draw the phase portrait for the system in this case.

(c) Find the critical point for $k = 0.5$. Determine its type and stability. Use the computer to draw the phase portrait for the system in this case.

(d) Find the value k_0 where the critical point changes its stability. Find the critical point and use the computer to draw the phase portrait for the system in this case.

(e) For $k > k_0$, the system exhibits a limit cycle. Determine its stability.

(f) Use the Runge-Kutta method to generate plots of $x(t)$ and $y(t)$. (You should choose t_f and the step size appropriately.)

(g) Estimate the period of the limit cycle.

(h) As k increases further, there is a value k_1 at which the limit cycle vanishes and the stability of the critical point again changes. Find k_1.

ANSWERS:

(a) When $dx/dt = 0$, $y = x^3/3 - x + k$. Substituting this expression for y into dy/dt yields $x^3 + 3/4x + 3k - 21/8$. The derivative of this function is $3x^2 + 3/4$ which is strictly positive for all x. Hence, $x^3 + 3/4x + 3k - 21/8 = 0$ at exactly one point x^*. Since we must also have $x^* + 0.8y - 0.7 = 0$, there is exactly one corresponding value y^*. Thus, regardless the value of k, there is only one fixed point.

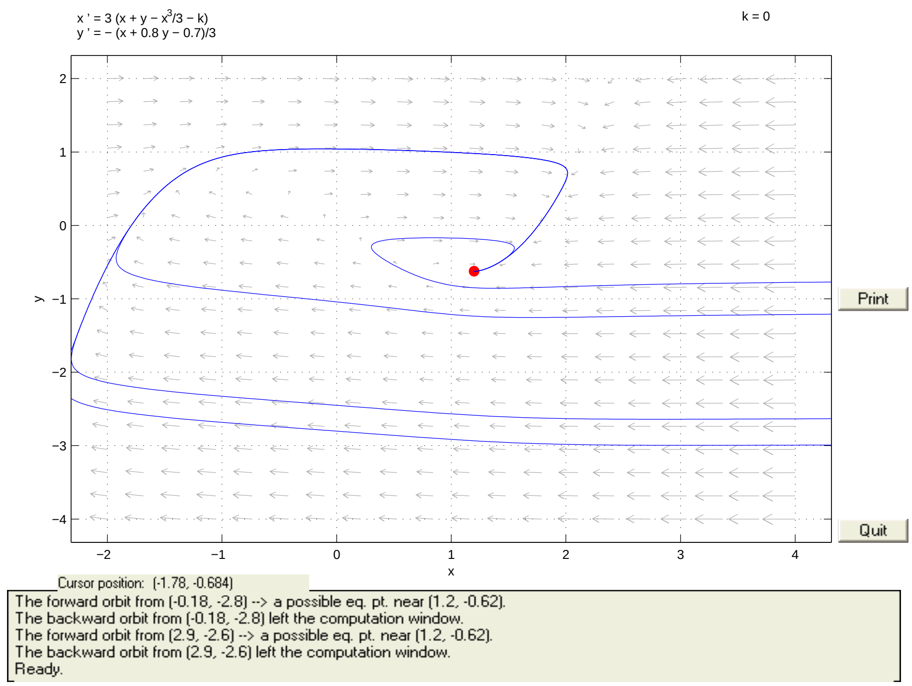
(b) We can use the MATLAB `roots` function to solve $x^3 + 3/4x - 21/8 = 0$. We obtain $x \approx 1.1994$ and $y \approx -0.62426$.

The Jacobian for the system is given by

$$J(x, y) = \begin{bmatrix} 3 - 3x^2 & 3 \\ -1/3 & -0.8/3 \end{bmatrix}$$

$$J(1.1994, -0.62426) = \begin{bmatrix} -1.31568 & 3 \\ -1/3 & -0.8/3 \end{bmatrix}$$

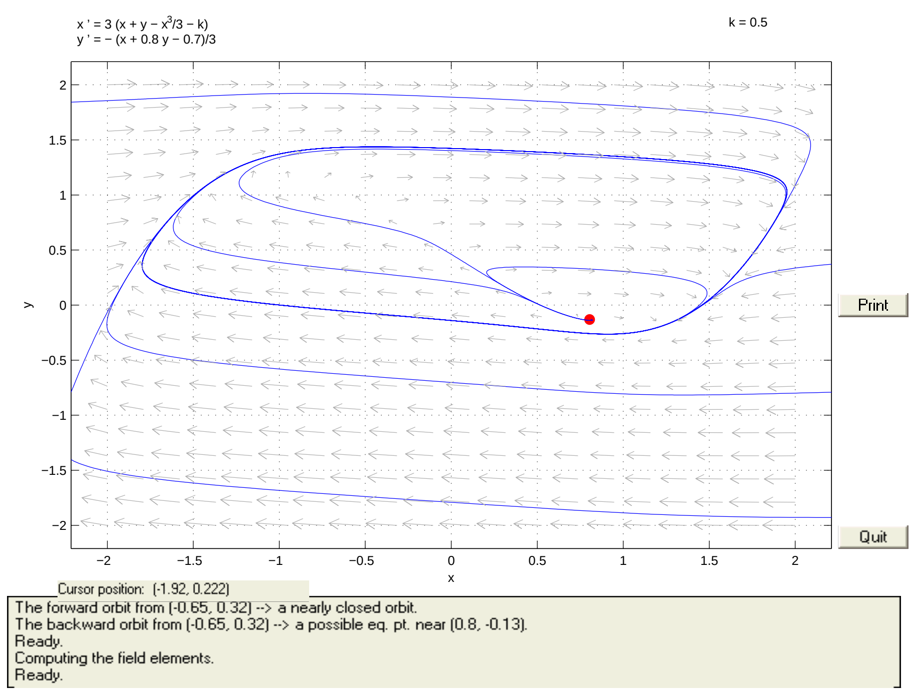
The evalues are complex conjugates with $Re(\lambda) < 0$ so the critical point is is a stable spiral.



(c) The critical point is (0.80485, -0.13106). The Jacobian at this point is

$$J(0.80485, -0.13106) = \begin{bmatrix} 1.056649 & 3 \\ -1/3 & -0.8/3 \end{bmatrix}$$

The evalues are complex conjugates with $Re(\lambda) > 0$ so the critical point is an unstable spiral.

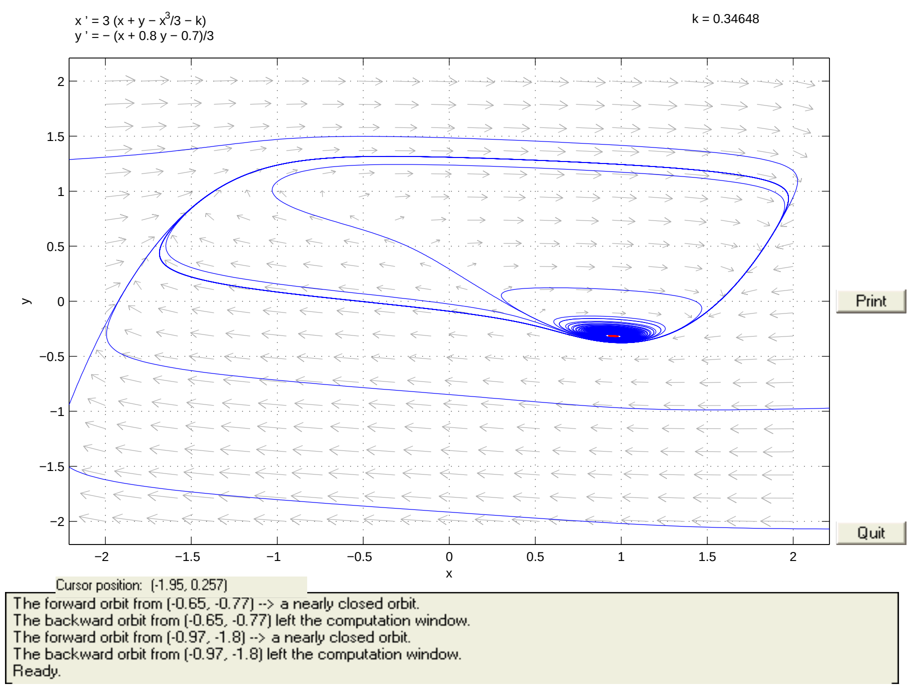


(d) We determine the evalues by solving

$$\begin{vmatrix} 3-3x^2-\lambda & 3 \\ -1/3 & -0.8/3-\lambda \end{vmatrix} = 0,$$

that is, $(3-3x^2-\lambda)(-0.8/3-\lambda)+1=0$. The bifurcation occurs when the roots are pure imaginary. This happens when $3x^2 = 3-0.8/3$, that is, when $x = \pm 0.95452$.

Using $x + 0.8y - 0.7 = 0$, we obtain two possible critical points: (0.95452, -0.31815) and (-0.95452, 2.06815). The associated values of k are obtained using $y - x^3/3 + x = k$. We obtain $k_0 = 0.34648$ and $k_1 = 1.40352$. (The value we want should be between 0 and 0.5.)

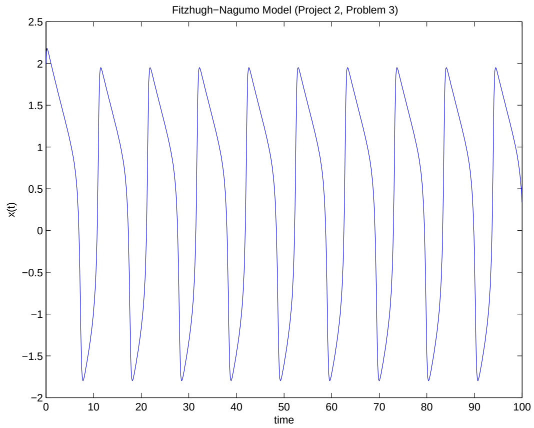


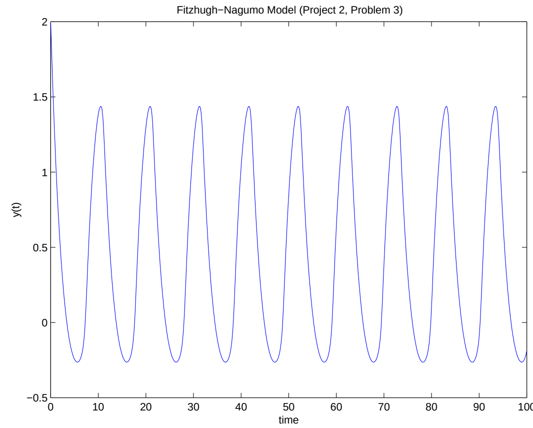
It is hard to tell from the phase portrait alone whether the fixed point is a center or spiral in this case.

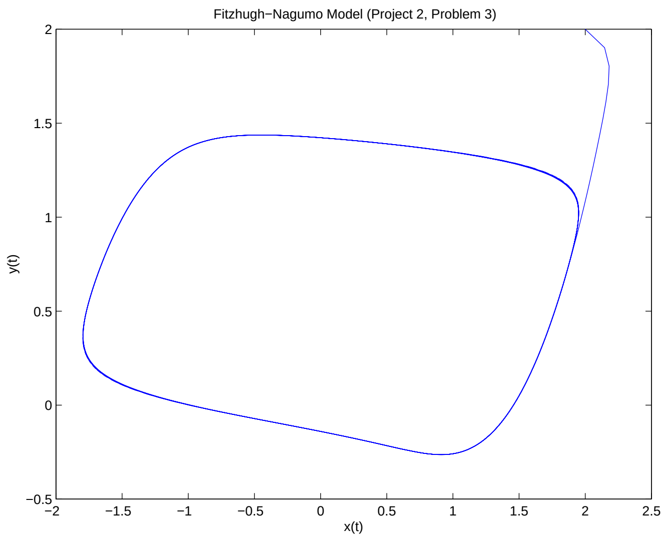
(e) Based on the plots, the limit cycle is stable.

(f) Use the Runge-Kutta method to generate plots of $x(t)$ and $y(t)$. (You should choose t_f and the step size appropriately.)

(g) Your answer here depends on the value of k that you used. Here are a few numbers: $k = 0.4$, $T \approx 11.23$; $k = 0.5$, $T \approx 10.37$; $k = 0.6$, $T \approx 9.93$. The figures below are for $k = 0.5$ with initial conditions $x(0) = y(0) = 2$.







(h) Again we want the value of k which causes the evalues to become pure imaginary – that would be $k_1 = 1.40352$ as computed in part (d).

4. **(30 pts).** Strogatz: Problem 7.5.6

ANSWERS:

We may write the second-order equation as a two-dimensional system of first order equations:

$$\dot{x} = y \qquad \dot{y} = a - \mu(x^2 - 1)y - x$$

(a) Every fixed point must have $y = 0$. Plugging this expression into $\dot{y}$, we see that there is only one fixed point: $(a, 0)$.

The Jacobian for the system is given by

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -2\mu xy - 1 & -\mu(x^2 - 1) \end{bmatrix}$$

$$J(a,0) = \begin{bmatrix} 0 & 1 \\ -1 & -\mu(a^2-1) \end{bmatrix}$$

The characteristic equation is

$$\lambda^2 + \mu(a^2-1)\lambda + 1 = 0$$

and the roots are

$$\lambda = \frac{-\mu(a^2-1) \pm \sqrt{\mu^2(a^2-1)^2 - 4}}{2}$$

- The roots are **real** if $|\mu(a^2-1)| > 2$. In this case, we would have $\sqrt{\mu^2(a^2-1)^2-4} < |\mu(a^2-1)|$ so the two roots will have the same sign and be distinct. Thus, in this case, there will a **stable node** if $|a| > 1$ and an **unstable node** if $|a| < 1$.
- The roots are **real, repeated** if $|\mu(a^2-1)| = 2$. There will be a stable (unstable) degenerate node if $|a| > 1$ ($|a| < 1$)
- The roots are **complex conjugates** if $|\mu(a^2-1)| < 2$. If $a^2 = 1$ then the roots will be pure imaginary (borderline case). If $|a| > 1$, there will be a stable spiral; if $|a| < 1$, there will be an unstable spiral.

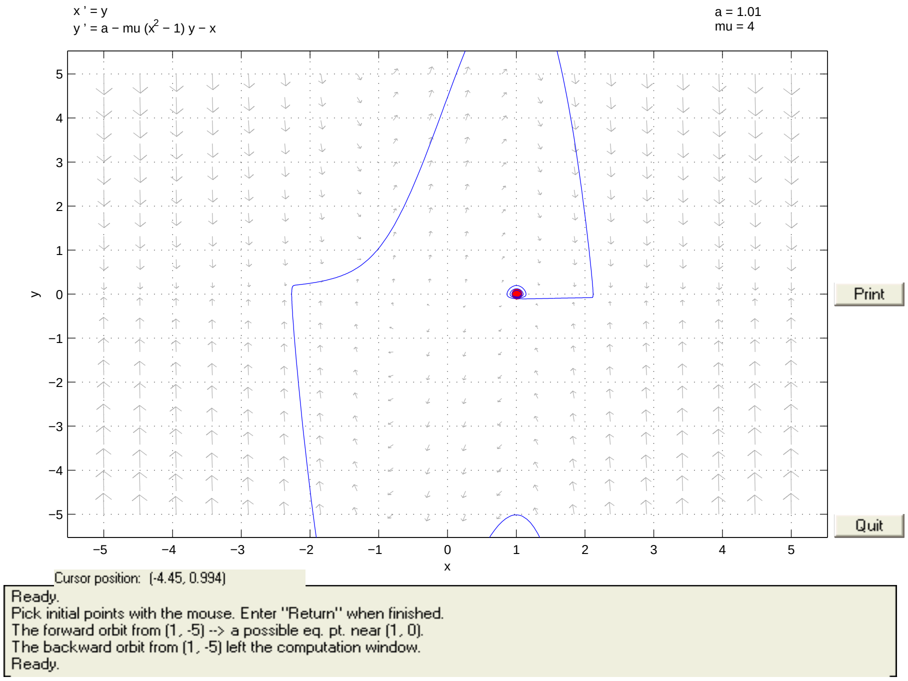
(b) In the Lienard plane, the system becomes

$$\dot{x} = \mu[z - F(x)] \qquad \dot{z} = \frac{1}{\mu}(a - x)$$

where $F(x) = 1/3x^2 - x$. The middle branch of the cubic nullcline occurs between the two extrema. Since $F'(x) = x^2 - 1 = 0$ when $x = \pm 1$, this is the area of instability for the fixed point as determined in part (a).

(c) The stable limit cycle follows the path (in the Lienard plane) comparable to that for the unbiased van der Pol equation. We see that $a_c = 1$.

(d) The figure below shows a trajectory passing through (1.01,-5). This trajectory illustrates the excitability of the system.



x ' = y
y ' = a − mu (x2 − 1) y − x
a = 1.01
mu = 4
Print
Quit
Cursor position: (-4.45, 0.994)
Ready.
Pick initial points with the mouse. Enter "Return" when finished.
The forward orbit from (1, -5) --> a possible eq. pt. near (1, 0).
The backward orbit from (1, -5) left the computation window.
Ready.