Dynamical Systems (550.391)
Take-Home Project II (Due: Monday, November 21, 2005)

General Directions: This project is open book, open notes. That is, you may use the course text and any notes from Fall 2005 in completing this project. You are to work individually. If you have any questions about the project, please contact Prof. Castello.

Show all work and document any assumptions to receive full credit on a problem. All problems are to be done by hand unless otherwise stated.

Please write neatly and clearly. If a problem has multiple parts, please list them in the correct order. Please staple your pages so that problems appear in the correct order.

1. **(40 pts).** A lake is initially stocked with 100 bass and 600 redear. There is ample food for the redear. Because bass prey on redear, the population of bass will increase at a rate, p, proportional to the number of encounters between the species; bass will also die at a rate, q, proportional to the bass population.

   The redear multiply at a rate, u, proportional to their population and die off at a rate, v, proportional to the number of encounters between the two species.

   Let $B(t)$ be the bass population at time t and let $R(t)$ be the redear population at time t.

   (a) Create a system of two first order ODEs to model the population dynamics.

   (b) Suppose it is know that $p = 0.00004$, $q = 0.02$, $u = 0.05$, and $v = 0.004$. Find the fixed points for this system and determine their type and stability.

   (c) Using the theory from Chapter 7, determine if the system has any limit cycles, or if it is possible to rule out periodic solutions.

   (d) Use the Runge-Kutta method to plot $R(t)$ and $B(t)$ for $0 < t < t_f$. (You should choose t_f and the step size appropriately.)

   (e) Based on the plots in part (d) estimate the period of oscillation for each species.

2. **(40 pts).** Consider the predator-prey model

$$\frac{dx}{dt} = x\left(1 - \frac{x}{7}\right) - \frac{6xy}{7 + 7x}, \qquad (1)$$

$$\frac{dy}{dt} = 0.2y\left(1 - \frac{Ny}{x}\right), \qquad (2)$$

   where N is a constant. The functions $x(t) \neq 0$ and $y(t)$ represent the populations of prey and predators respectively.

   (a) How does this model differ from the predator-prey models we have studied in the past?

(b) In the context of population modeling, interpret the terms

(i) $x(1 - \frac{x}{7})$

(ii) $-\frac{6xy}{7+7x}$

(ii) $0.2y\left(1 - \frac{Ny}{x}\right)$

(c) Analyze the population dynamics when $N = 2.5$. That is, determine the type and stability of any fixed points; determine the stability of any limit cycles; sketch or computer-generate a phase plane portrait.

(d) Use your analysis from part (c) to provide discuss the population dynamics for this system when $N = 2.5$.

(e) Analyze the population dynamics when $N = 0.5$.

(f) Use your analysis from part (e) to provide discuss the population dynamics for this system when $N = 0.5$.

(g) (Extra Credit!) Create a trapping region for the system when $N = 0.5$ and use the Poincare-Bendixson Theorem to prove the existence of a limit cycle.

3. **(40 pts).** The system

$$\frac{dx}{dt} = 3(x + y - \frac{x^3}{3} - k), \qquad (3)$$

$$\frac{dy}{dt} = -\frac{1}{3}(x + 0.8y - 0.7), \qquad (4)$$

is a special case of the *Fitzhugh-Nagumo* equations, which model the transmissions of neural impulses along an axon. The parameter k is known as the *external stimulus.*

(a) Show that system has only one critical point regardless of the value of k.

(b) Find the critical point for $k = 0$. Determine its type and stability. Use the computer to draw the phase portrait for the system in this case.

(c) Find the critical point for $k = 0.5$. Determine its type and stability. Use the computer to draw the phase portrait for the system in this case.

(d) Find the value k_0 where the critical point changes its stability. Find the critical point and use the computer to draw the phase portrait for the system in this case.

(e) For $k > k_0$, the system exhibits a limit cycle. Determine its stability.

(f) Use the Runge-Kutta method to generate plots of $x(t)$ and $y(t)$. (You should choose t_f and the step size appropriately.)

(g) Estimate the period of the limit cycle.

(h) As k increases further, there is a value k_1 at which the limit cycle vanishes and the stability of the critical point again changes. Find k_1.

4. **(30 pts).** Strogatz: Problem 7.5.6