Dynamical Systems (550.391)
Take-Home Project I (Due: Thursday, October 13, 2005)

General Directions: This project is open book, open notes. That is, you may use the course text and any notes from Fall 2005 in completing this project. You are to work individually. If you have any questions about the project, please contact Prof. Castello.

Show all work and document any assumptions to receive full credit on a problem. All problems are to be done by hand unless otherwise stated.

1. **(20 pts)** Strogatz: Problem 2.3.4
   Answer:

   (a) We need to maximize the function

   $$\frac{1}{N}\frac{dN}{dt} = r - a(N-b)^2 = f(N)$$

   Since $f'(N) = -2a(N-b)$ and $f''(N) = -2a$, the function will have a maximum if $-2a < 0$, so we need $a > 0$. Next, the function achieves an optimum when $f'(N) = 0$; this happens when $N = b$. Since N is a population size, we must have $b \geq 0$. Since we want the maximum to occur at an intermediate value, we need $0 < b < \infty$.

   Now the tricky part is determining a good range for r. When $N = b$, $f(N) = r$. If this is the maximum effective growth rate, we clearly need $r > 0$. But what we also need is a positive effective growth when $N \approx 0$; otherwise the population will die out before it can reach the maximum growth rate r. This means we need $r - a(0-b)^2 > 0$, i.e., $r > ab^2$.

   (b) To find the fixed points, we now put the equation in *traditional form*:

   $$\frac{dN}{dt} = N(r - a(N-b)^2) = \tilde{f}(N)$$

   The fixed points occur when $N_1 = 0$, $N_2 = b + \sqrt{r/a}$ and $N_3 = b - \sqrt{r/a}$. But observe that with $r > ab^2$, $\sqrt{r/a} > b$, so $N_3 < 0$ which makes no physical sense. Hence, there are only two nonnegative fixed points.

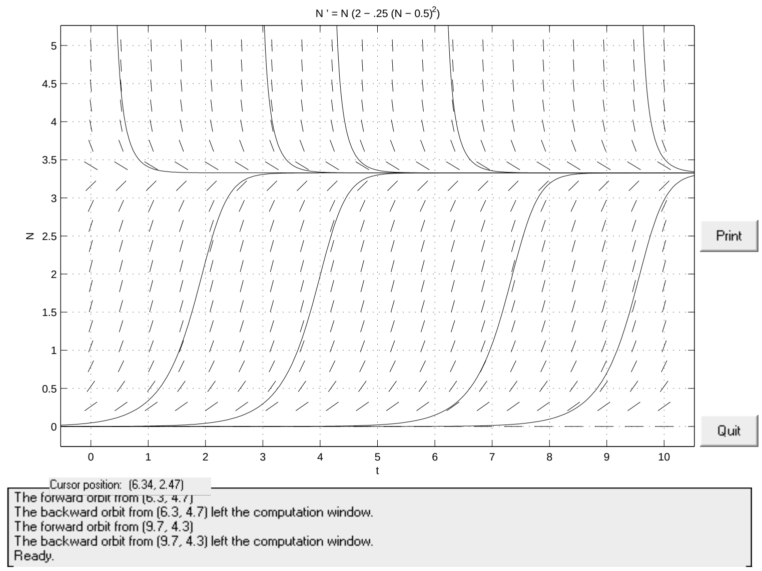
   We may use $\tilde{f}'(N) = -2aN(N-b) + r - a(N-b)^2$ to determine the stability of the fixed points. For $N_1 = 0$, we have $\tilde{f}'(0) = r + ab^2 > 2r$ (since $r > ab^2$). Hence $N_1 = 0$ is *unstable*.

   For N_2, we have

   $$\tilde{f}'(N_2) = -2b\sqrt{ar} - r - a < 0$$

   since all parameters are strictly positive. Hence N_2 is *stable*.

(c) The figure below uses $a = 0.25$, $b = 0.5$, and $r = 2$.



(d) The trajectories are very much like those exhibited by the logistic equation.

2. **(50 pts)** Strogatz: Problem 3.7.6 (a)-(j)
Answer:

(a) We have

$$\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0$$

hence

$$x + y + z = constant$$

and we may call this constant N.

(b)

$$\begin{aligned}
\frac{dx}{dz} &= \frac{-kxy}{ly} = -\frac{kx}{l} \\
\frac{dx}{x} &= -\frac{k}{l}dz \\
\ln x &= -\frac{k}{l}z + C \\
\ln x_0 &= -\frac{k}{l}(0) + C = C \\
\ln x &= -\frac{k}{l}z + \ln x_0 \\
x &= x_0 e^{-kz/l}
\end{aligned}$$

(c)

$$\begin{aligned}\dot{z} &= ly = l(N - x - z) \\ &= l(N - z - x_0 e^{-kz/l})\end{aligned}$$

(d) Let

$$u = \frac{k}{l}z \quad \Rightarrow \quad \frac{du}{dt} = \frac{k}{l}\frac{dz}{dt}$$

$$\begin{aligned}\frac{dz}{dt} &= l(N - z - x_0 e^{-u}) \\ \frac{1}{lx_0}\frac{dz}{dt} &= \frac{1}{x_0}(N - z) - e^{-u} \\ \frac{1}{lx_0}\frac{l}{k}\frac{du}{dt} &= \frac{1}{x_0}(N - \frac{l}{k}u) - e^{-u} \\ \frac{1}{kx_0}\frac{du}{dt} &= \frac{1}{x_0}N - \frac{l}{kx_0}u - e^{-u}\end{aligned}$$

If we let $a = N/x_0$, $b = l/kx_0$ and $\tau = kx_0 t$, we obtain

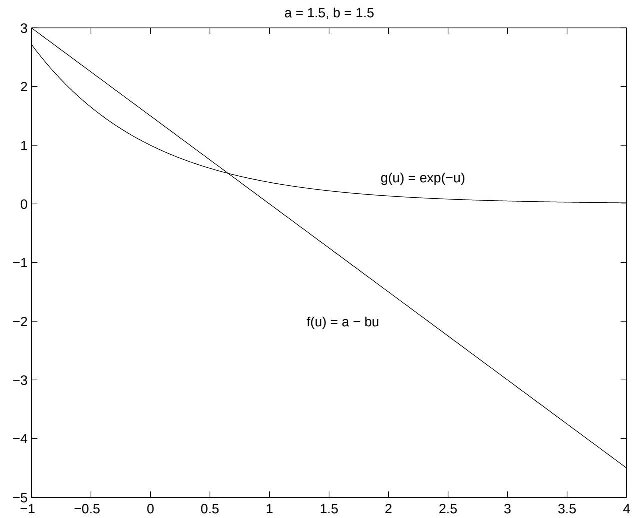
$$\frac{du}{d\tau} = a - bu - e^{-u}$$

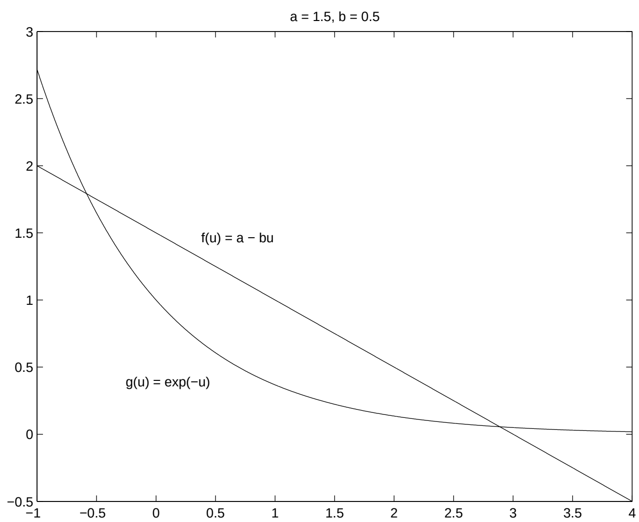
which is the desired result.

(e) We have defined $a = N/x_0$. Since N is the total population and x_0 is the number of healthy people at time $t = 0$, we must have $N \geq x_0$. Hence $a \geq 1$.

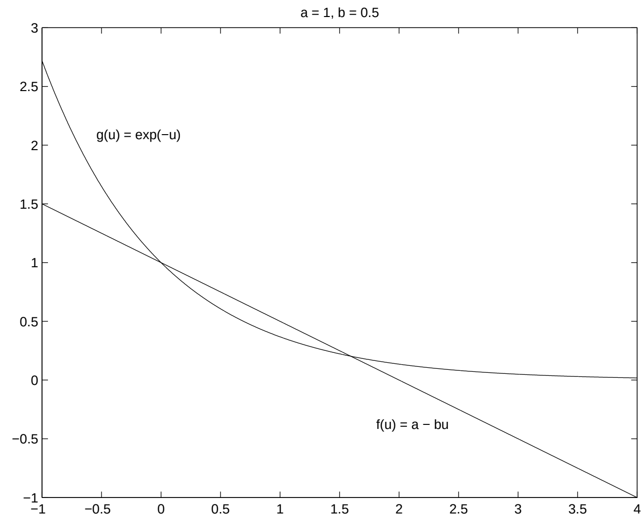
We have defined $b = l/kx_0$, with $l, k > 0$ (given). In addition, the number of people initially healthy should be strictly positive. Hence $b > 0$.

(f) We may find the fixed points graphically, restricting $a \geq 1$ and $b > 0$. When $a > 1$ the figures below show that there is only one intersection point $u = u^*$ where $u^* > 0$, regardless of the value of b. In this case, the fixed point is *stable*.

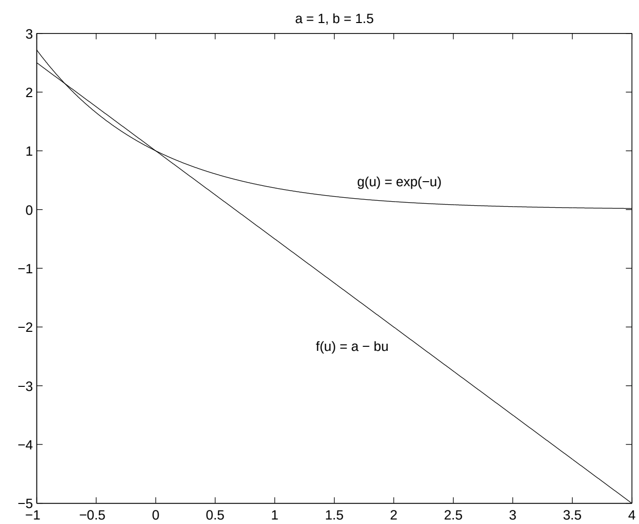




When $a = 1$ and $0 < b < 1$, the figure below shows that there are two intersection points: $u = 0$ and $u = u^*$ where $u^* > 0$. In this case, the fixed point $u = 0$ is *unstable* and the fixed point $u = u^*$ is *stable*.



When $a = 1$ and $b > 1$, the figure below shows that the only intersection point is at $u = 0$. In this case, the fixed point $u = 0$ is *stable*.



(g) We see

$$\frac{dz}{dt} = \frac{l}{k}\frac{du}{dt} = ly.$$

Since all the functions are constant multiples of each other and all the constants are strictly positive, the functions $dz/\text{d}t$, du/dt, and y achieve their optima at the same time.

(h) The function $f(u) = a - bu - e^{-u}$ is maximized when $df/du = -b + e^{-u} = 0$ (since $f''(u) = -e^{-u} < 0$ for all u). Thus, this function is concave, and if $df/du = 0$ when $u = \hat{u}$, then $f(u)$ is increasing for all $u < \hat{u}$ and decreasing for all $u > \hat{u}$. Clearly $\hat{u} = -\ln b = \ln(1/b)$. For this to make physical sense, $\ln(1/b) > 0$, so $b < 1$.

(i) Recall that $z(0) = 0$ (no dead people) so $u(0) = 0$. Hence, if $b > 1$, $df/du = -b + e^{-u} < 0$ for all $u > 0$; i.e., $f(u) = du/d\tau$ is decreasing from $u(0) = 0$.

(j) When $b = 1$, the date rate equals the infection rate times number of initially healthy people. When $b < 1$, the sick people die before infecting many others. When $b > 1$, the sick people infect many others before they die, causing an epidemic.

3. **(15 pts)** Consider the linear system

$$\frac{dx}{dt} = hx - 4y; \qquad \frac{dy}{dt} = x + hy$$

where h is constant. Discuss the effect of the value of h on the type and stability of the critical point (0,0).
Answer:
The characteristic equation is $\lambda^2 - 2h\lambda + h^2 - 4 = 0$. $\tau = 2h, \Delta = h^2 + 4 > 0$.
$\tau^2 - 4\Delta = -16 < 0$.
When $\tau < 0$, that is $h < 0$, $(0,0)$ is an stable spiral.
When $\tau = 0$, that is $h = 0$, $(0,0)$ is a center.
When $\tau > 0$, that is $h > 0$, $(0,0)$ is a unstable spiral.

4. **(15 pts)** Consider the linear system

$$\frac{dx}{dt} = -x - hy; \qquad \frac{dy}{dt} = x - y$$

where h is constant. Discuss the effect of the value of h on the type and stability of the critical point (0,0).
Answer:
The characteristic equation is $\lambda^2 + 2\lambda + (h+1) = 0$. $\tau = -2 < 0, \Delta = h + 1$.
If $h < -1$, $\Delta < 0$. $(0,0)$ is a saddle point.
If $h = -1$, $\Delta = 0$. $(0,0)$ is an nonisolated fixed point.
If $-1 < h < 0$, $\Delta > 0, \tau < 0, \tau^2 - 4\Delta > 0$. $(0,0)$ is a stable node.
If $h = 0$, $\Delta > 0, \tau = 0$. $(0,0)$ is a center.
If $h > 0$, $\Delta > 0, \tau < 0, \tau^2 - 4\Delta < 0$. $(0,0)$ is a stable spiral.

5. **(20 pts)** This problem deals with the almost linear system

$$\frac{dx}{dt} = y + hx(x^2 + y^2) \qquad \frac{dy}{dt} = -x + hy(x^2 + y^2)$$

(a) What is the type and stability of the critical point (0,0) when $h = 0$?

(b) Suppose that $h \neq 0$. Let $r^2 = x^2 + y^2$. Show that $dr/dt = hr^3$.

(c) Suppose that $h = -1$. Integrate the differential equation in (b). What is $\lim_{t\to\infty} r(t)$? What can you say about the type and stability of the critical point (0,0) in this case?

(d) Suppose that $h = 1$. Integrate the differential equation in (b). What is $\lim_{t\to\infty} r(t)$? What can you say about the type and stability of the critical point (0,0) in this case?

Answer:

(a) When $h = 0$,

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -x.$$

The characteristic equation is $\lambda^2 + 1 = 0$. $\tau = 0, \Delta = 1 > 0$. So $(0,0)$ is a center.

(b) $x = r\cos\theta, y = r\sin\theta$.

$$\begin{cases} dx &= \cos\theta dr - r\sin\theta d\theta \\ dy &= \sin\theta dr + r\cos\theta d\theta \end{cases} \Rightarrow \begin{cases} dr &= \cos\theta dx + r\sin\theta dy \\ d\theta &= \frac{\cos\theta dy - \sin\theta dx}{r} \end{cases}$$

So

$$\dot{r} = \cos\theta(r\sin\theta + hr\cos\theta r^2) + \sin\theta(-r\cos\theta + hr\sin\theta r^2) = hr^3.$$

$$\dot{\theta} = \frac{\cos\theta(-r\cos\theta + hr\sin\theta r^2) - \sin\theta(r\sin\theta + hr\cos\theta r^2)}{r} = -1.$$

(c) If $h = -1$, then $\dot{r} = -r^3 \Rightarrow 2r^2 = 1/(t + C)$, where C is a positive constant. So when $t \to +\infty, r \to 0$. In this case, the critical point $(0,0)$ is a stable spiral.

(d) If $h = -1$, then $\dot{r} = r^3 \Rightarrow 2r^2 = 1/(C - t)$, where C is a positive constant. So when $t \to -\infty, r \to 0$. In this case, the critical point $(0,0)$ is an unstable spiral.

6. **(30 pts)** Find all critical points for each of the given systems, and investigate the type and stability of each. Construct a phase portrait for each system using a computer system or graphing calculator.

(a)

$$\frac{dx}{dt} = y^2 - 1 \qquad \frac{dy}{dt} = x^3 - y$$

(b)

$$\frac{dx}{dt} = xy - 2 \qquad \frac{dy}{dt} = x - 2y$$

Answer:

(a) $\begin{cases} y^2 - 1 &= 0 \\ x^3 - y &= 0 \end{cases} \Rightarrow (x, y) = (1, 1)$ or $(-1, -1)$.

The Jacobian is $J = \begin{pmatrix} 0 & 2y \\ 3x^2 & -1 \end{pmatrix}$.

If $(x, y) = (1, 1)$, $J = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$.

$\Delta = -6 < 0$. So $(1, 1)$ is a saddle point of the linearized system. Since saddle point is not the borderline case, $(1, 1)$ is also a saddle point of the original system.

If $(x,y)=(-1,-1)$, $J=\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$.

$\tau=-1<0, \Delta=6>0, \tau^2<4\Delta$. So $(-1,-1)$ is a stable spiral of the linearized system. Since spiral is not the borderline case, $(-1,-1)$ is also a stable spiral of the original system.

For phase portrait for original system, see figure 1.

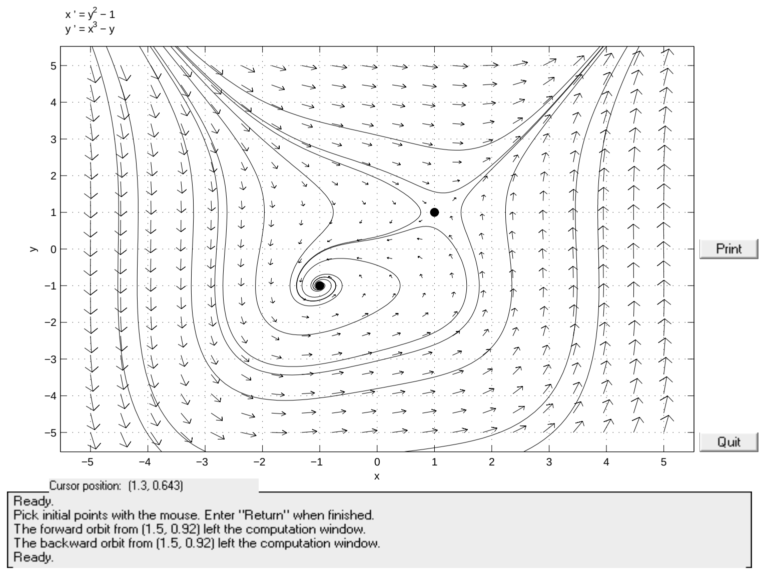


Figure 1

(b) $\begin{cases} xy-2 &= 0 \\ x-2y &= 0 \end{cases} \Rightarrow (x,y)=(2,1) \text{ or } (-2,-1).$

The Jacobian is $J=\begin{pmatrix} y & x \\ 1 & -2 \end{pmatrix}$.

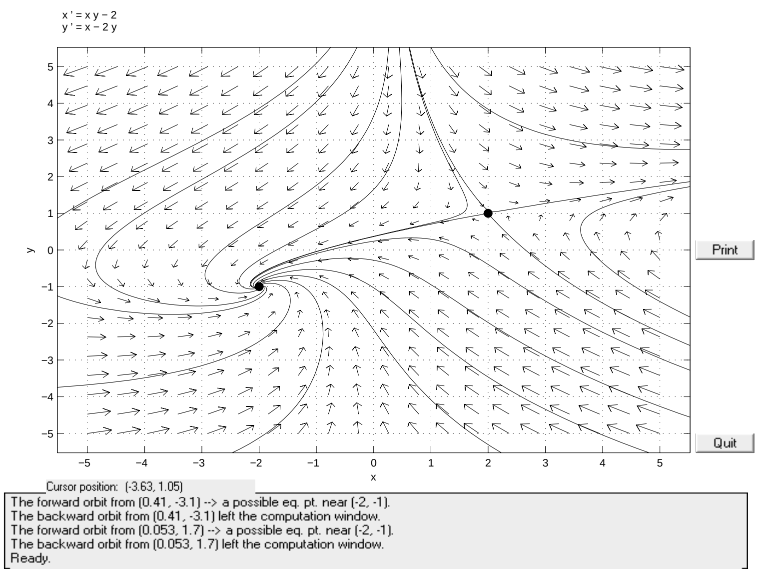
If $(x,y)=(2,1)$, $J=\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$.

$\Delta=-4<0$. So $(2,1)$ is a saddle point of the linearized system. Since saddle point is not the borderline case, $(2,1)$ is also a saddle point of the original system.

If $(x,y)=(-2,-1)$, $J=\begin{pmatrix} -1 & -2 \\ 1 & -2 \end{pmatrix}$.

$\tau=-3<0, \Delta=4>0, \tau^2<4\Delta$. So $(-2,-1)$ is a stable spiral of the linearized system. Since spiral is not the borderline case, $(-2,-1)$ is also a stable spiral of the original system.

For phase portrait for original system, see figure 2.



x ' = x y − 2
y ' = x − 2 y
y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
Print
Quit
Cursor position: (-3.63, 1.05)
The forward orbit from (0.41, -3.1) --> a possible eq. pt. near (-2, -1).
The backward orbit from (0.41, -3.1) left the computation window.
The forward orbit from (0.053, 1.7) --> a possible eq. pt. near (-2, -1).
The backward orbit from (0.053, 1.7) left the computation window.
Ready.

Figure 2