# Dynamical Systems(550.391)
# Homework 8 (Due Monday, December 05, 2005)

**General Directions:** You must show all work and document any assumptions to receive full credit on a problem. Feel free to use MATLAB or any other computer system to find the fixed points, eigenvalues, and, if necessary, eigenvectors for your analysis. (Refer to the Software Usage Guidelines on the course website to determine what computer output you need to submit.)

1. Strogatz: Problem 8.3.2

   Answer:

   **(a)**

   $$\lim_{x\to\infty}\frac{\dot{y}}{\dot{x}} = \lim_{x\to\infty}\frac{b-x^2y}{a-x+x^2y} = -1.$$

   So the vector field at large x is roughly parallel to the line $y = -x$.
   $\dot{x} - (-\dot{y}) = a + b - x$. Hence $-\dot{y} > \dot{x}$ if $x > a + b$.
   Therefore, we have the following trapping region:
   $\{(x, y)|x \geq a, y \geq 0, y \leq \frac{b}{a^2}, x + y \leq a + b + \leq \frac{b}{a^2}\}$.

   **(b)**

   $$\left\{\begin{array}{lcl} a - x + x^2y & = & 0 \\ b - x^2y & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} x & = & a+b \\ y & = & \frac{b}{(a+b)^2} \end{array}\right.$$

   So there is only one fixed point: $(a + b, \frac{b}{(a+b)^2})$.

   $$J(a+b, \frac{b}{(a+b)^2}) = \begin{pmatrix} \frac{b-a}{a+b} & (a+b)^2 \\ \frac{2b}{a+b} & -(a+b)^2 \end{pmatrix}$$

   $\tau = \frac{b-a-(a+b)^3}{a+b}, \Delta = (a+b)^2 > 0$.
   When $\tau > 0$, it's unstable node($\tau^2 > 4\Delta$); unstable star/degenerate node($\tau^2 = 4\Delta$); or unstable spiral($\tau^2 < 4\Delta$).
   When $\tau = 0$, it's center or spiral.
   When $\tau < 0$, it's stable node($\tau^2 > 4\Delta$); stable star/degenerate node($\tau^2 = 4\Delta$); or stable spiral($\tau^2 < 4\Delta$).

   **(c)** The eigenvalues are the roots of $\lambda^2 - \tau\lambda + \Delta^2 = 0$. So $\lambda = \frac{\tau\pm\sqrt{\tau^2-4\Delta}}{2}$.
   When $b - a = (a + b)^3$, $\tau = 0$. Since $\Delta > 0$, $\lambda = \pm\sqrt{\Delta}i$. So when $b - a - (a + b)^3$ varies from negative values to positive values, the eigenvalues cross the imaginary axis into the right half-plane. Thus, the system undergoes a Hopf bifurcation.

   **(d)** It's a supercritical bifurcation.

   **(e)** The stability diagram is figure 1.

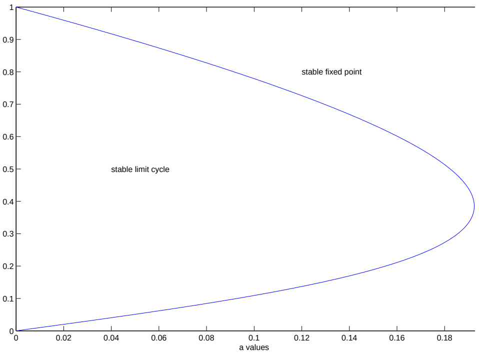

Figure 1

2. Plot a bifurcation diagram for the system

$$\dot{r} = r\left((r-1)^2 - \mu r\right) \qquad \dot{\theta} = 1.$$

Give a possible reason as to why this type of bifurcation diagram should be known as a *fold bifurcation*.
*Comment:* You'll have to convert back to Cartesian coordinates.

Answer:
Since $\theta = 1$, the only fixed point is $r = 0$. The limit cycle satisfies $\dot{r} = 0$, so the limit cycle is $r = r_0$, where $r_0$ is the positive root of $(r-1)^2 - \mu r = 0$.
When $\mu > 0$, $(r-1)^2 - \mu r = 0$ has two positive roots. One represents a stable limit cycle, the other represents an unstable limit cycle.
When $\mu \leq 0$, $(r-1)^2 - \mu r = 0$ doesn't have positive roots. So there are no limit cycles in this case.

The bifurcation diagram is figure 2.

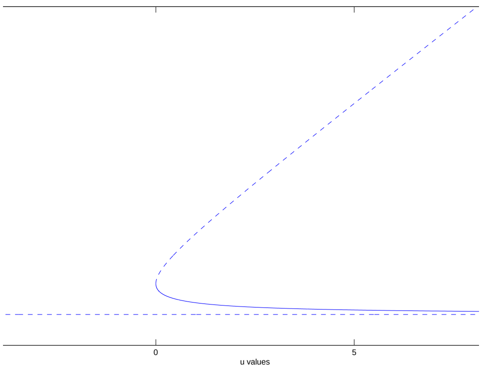

Figure 2

The bifurcation is called a fold bifurcation because two limit cycles coalesce and annihilate.

3. Plot a bifurcation diagram for the system

$$\dot{r} = r\left(\mu - 0.2r^6 + r^4 - r^2\right) \qquad \dot{\theta} = -1$$

and indicate the regions where the system is multistable and/or possibly bistable.

*Comment:* You'll have to convert back to Cartesian coordinates.

Answer:
Since $\theta = -1$, the only fixed point is $r = 0$. The limit cycle satisfies $\dot{r} = 0$, so the limit cycle is $r = r_0$, where $r_0$ is the positive root of $\mu - 0.2r^6 + r^4 - r^2 = 0$.
When $\mu < -0.6537$, $\mu - 0.2r^6 + r^4 - r^2 = 0$ have no positive roots. So there are no limit cycles in this case.
When $\mu = -0.6537$, $\mu - 0.2r^6 + r^4 - r^2 = 0$ has one positive root. It represents a half-stable limit cycle.
When $-0.6537 < \mu \leq 0$, $\mu - 0.2r^6 + r^4 - r^2 = 0$ has two positive roots. One represents a stable limit cycle, the other represents an unstable limit cycle.
When $0 < \mu < 0.2833$, $\mu - 0.2r^6 + r^4 - r^2 = 0$ has three positive roots. One represents an unstable limit cycle, the others represent two stable limit cycles.
When $\mu = 0.2833$, $\mu - 0.2r^6 + r^4 - r^2 = 0$ has two positive roots. One represents a stable limit cycle, the other represents a half-stable limit cycles.
When $\mu > 0.2833$, $\mu - 0.2r^6 + r^4 - r^2 = 0$ has one positive root. It represents a stable limit cycle.

The bifurcation diagram is figure 3.

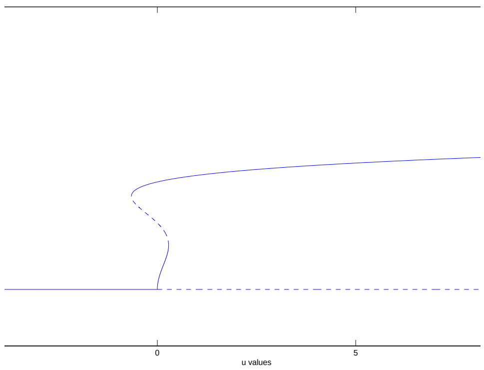

Figure 3

4. Find and classify all fixed points of the system

$$\begin{aligned}\dot{x} &= x + 2z\\ \dot{y} &= y - 3z\\ \dot{z} &= 2y + z.\end{aligned}$$

Answer:

$$\left\{\begin{array}{rcl} x+2z & = & 0 \\ y-3z & = & 0 \\ 2y+z & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} x & = & 0 \\ y & = & 0 \\ z & = & 0 \end{array}\right.$$

So the only fixed point is (0,0,0).
The characteristic equation is:

$$\left|\begin{array}{ccc} \lambda-1 & 0 & -2 \\ 0 & \lambda-1 & 3 \\ 0 & -2 & \lambda-1 \end{array}\right| = 0$$

The eigenvalues are $\lambda = 1, 1 \pm \sqrt{6}i$. Let the corresponding eigenvector be $x_1, x_2, x_3$. $1 \pm \sqrt{6}i$ gives a unstable spiral on the plane spanned by $x_2, x_3$, 1 means the trajectory tends to infinity at the direction of $x_3$, so (0,0,0) is a unstable fixed point.

5. A three-dimensional Lotka-Volterra model is given by

$$\begin{array}{rcl} \dot{x} & = & x(1-2x+y-5z) \\ \dot{y} & = & y(1-5x-2y-z) \\ \dot{z} & = & z(1+x-3y-2z). \end{array}$$

**(a)** Identify the fixed points of the system that are in the first quadrant.

**(b)** Show that $x+y+z=0.5$ is a solution to this system.

**(c)** Substitute $z = 0.5-(x+y)$ into the system to derive a two dimensional model in $x$ and $y$. Create a phase portrait in $x, y$-space and interpret what you observe in the context of a population model.

**(d)** Repeat part(c) but substitute for $x$ and then $y$.

Answer:

**(a)** In the first quadrant, $x>0, y>0, z>0$, so

$$\left\{\begin{array}{rcl} \dot{x} & = & 0 \\ \dot{y} & = & 0 \\ \dot{z} & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} 1-2x+y-5z & = & 0 \\ 1-5x-2y-z & = & 0 \\ 1+x-3y-2z & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} x & = & \frac{1}{14} \\ y & = & \frac{3}{14} \\ z & = & \frac{3}{14} \end{array}\right.$$

**(b)** When $x+y+z=0.5$,
$\dot{z} = z(1+x-3y-2z) = (0.5-x-y)(1+x-3y-2z-1+2x+2y) = -3x^2+y^2-2xy+1.5x-0.5y$.
$\dot{x}+\dot{y} = x(1-2x+y-5z)+y(1-5x-2y-z) = 3x^2-y^2+2xy-1.5x+0.5y$.
So $\dot{x}+\dot{y}+\dot{y}=0$.

**(c)** $\dot{x} = x(1-2x+y-2.5+5x+5y) = 3x^2+6xy-1.5x$.
$\dot{y} = y(1-5x-2y-0.5+x+y) = -y^2-4x^y+0.5y$.
So the two-dimensional model is

$$\left\{\begin{array}{rcl} \dot{x} & = & 3x^2+6xy-1.5x \\ \dot{y} & = & -y^2-4x^y+0.5y \end{array}\right.$$

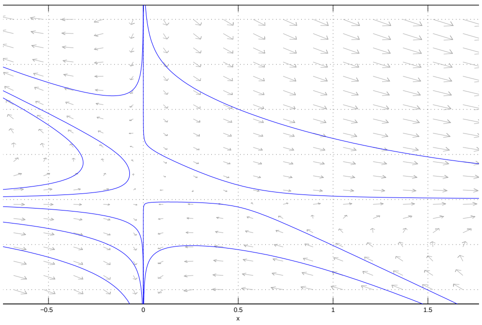

Figure 4

Its phase portrait is figure 4

**(d)** Similar to part(c), the two dimensional models are:

$$\begin{cases} \dot{y} &= 3y^2 + 4yz - 1.5y \\ \dot{y} &= -3z^2 - 4yz + 1.5z \end{cases} \quad \begin{cases} \dot{x} &= -3x^2 - 6xz + 1.5x \\ \dot{z} &= z^2 + 4xz - 0.5z \end{cases}$$

Their phase portraits are figure 5 and 6.

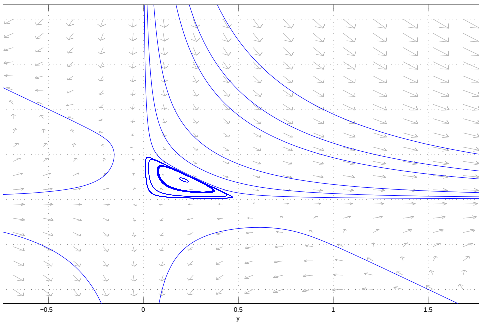

Figure 5

6. Select one of the following dynamical systems for study for your final project (content details forthcoming; the required articles for most of the topics are available on the course website)

   **(a)** Chua's Circuit (for background, see L.P. Shil'nikov, "Chua's circuit: Rigorous results and future problems," *International Journal of Bifurcation Chaos*, 4(1994), 489-519, although a different article may be required.)

   **(b)** Belousov-Zhabotinsky Chemical Reaction (see Sec 12.4 in text for an overview)

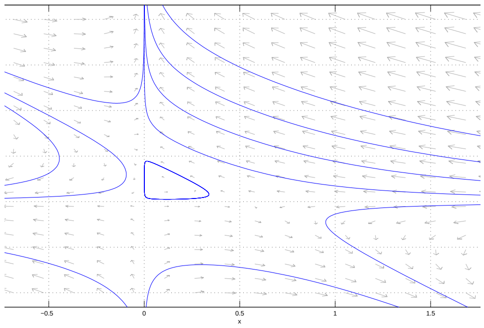

Figure 6

**(c)** Rikitake's model of geomagnetic reversals (See Exercise 9.2.6)

**(d)** Chen's System

**(e)** SIQR model for childhood diseases

**(f)** Infectious disease model

**(g)** Forest Dynamics

**(h)** Laser model

**(i)** Modified Lorenz System

**(j)** Rossler System (See Sec 12.3 for overview)