Dynamical Systems (550.391)
Homework 7 (Due Thursday, November 10, 2005)

General Directions: You must show all work and document any assumptions to receive full credit on a problem. Feel free to use MATLAB or any other computer system to assist your analysis. (Refer to the Software Usage Guidelines on the course website to determine what computer output you need to submit.)

There is an example Lienard plane analysis available on the course website.

1. Strogatz: Problem 7.3.4
   Answer:

   (a) The Jacobian is $J = \begin{pmatrix} 1-12x^2-y^2-\frac{y}{2} & -\frac{1}{2}-2xy-\frac{x}{2} \\ 2-8xy+4x & 1-4x^2-3y^2 \end{pmatrix}$.

   $J(0,0) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 1 \end{pmatrix}$. $\tau = 2, \Delta = 2$. So origin is an unstable spiral.

   (b) $V = (1-4x^2-y^2)^2 > 0$. $\frac{\partial V}{\partial x} = -16x(1-4x^2-y^2)$. $\frac{\partial V}{\partial y} = -4y(1-4x^2-y^2)$.
   $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = -4(4x^2+y^2)(4x^2+y^2-1)^2 < 0$.
   $V > 0, \dot{V} < 0 \Rightarrow V \to 0$ when $t \to \infty$.
   $V \to 0 \Rightarrow 4x^2+y^2 \to 1$. So all trajectories approaches the ellipse $4x^2+y^2 = 1$ as $t \to \infty$.

2. Strogatz: Problem 7.4.2
   Answer:

   (a) When $\mu > 0$, let $f(x) = \mu(x^4-1), g(x) = x$. Then

   (1) $f(x)$ and $g(x)$ are continuously differentiable for all x;
   (2) $g(-x) = -x = -g(x)$;
   (3) $g(x) = x > 0$ for all $x > 0$;
   (4) $f(-x) = \mu(x^4-1) = f(x)$;
   (5) $F(x) = \int_0^x f(u)du = \mu(\frac{x^5}{5} - x)$. $F(x)$ has exactly one positive zero at $x = \sqrt[4]{5}$, is negative for $0 < x < \sqrt[4]{5}$, is positive for $x > \sqrt[4]{5}$, and $F(x) \to \infty$ as $x \to \infty$.

   By Lienard's Theorem, the system has a unique stable limit cycle.

   (b) The phase portrait for the case $\mu = 1$ is as Figure 1.

   (c) When $\mu < 0$, the first four conditions in the *Liénard'sTheorem* is satisfied, that is:

   (1) $f(x)$ and $g(x)$ are continuously differentiable for all x;
   (2) $g(-x) = -x = -g(x)$;
   (3) $g(x) = x > 0$ for all $x > 0$;

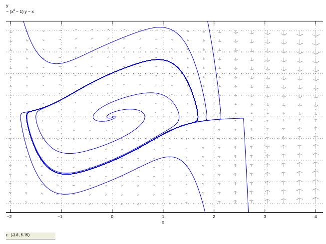

Figure 1:

(4) $f(-x) = \mu(x^4 - 1) = f(x)$;

But the fifth condition fails. Specifically, suppose $F(x) = \int_0^x f(u)du = \mu(\frac{x^5}{5} - x)$. $F(x)$ is positive at small $|x|$ and negative at large $|x|$. That means small oscillations are damped down and large oscillations are pumped up. So the system will have an unstable limit cycle.

3. Strogatz: Problem 7.5.2
   Answer:
   Advantage of *Liénard* plane:

   (1) In the *Liénard* plane, the limit cycle converges to a fixed shape as $\mu \to \infty$; that's not true in the usual phase plane.

   (2) In the standard phase plane, if we look at vector field around the trajectory, we can see that the vectors are large in the middle of the trajectory and small near its ends. This is the only clue as to show how fast you are moving. The nice thing about the *Liénard* plane is that it show you how your moving without the vector field.

4. Strogatz: Problem 7.5.3
   Answer:
   Let $F(x) = \int (x^2 - 4)dx = x^3/3 - 4x, y = \frac{1-x}{k}$. Then

$$\begin{aligned} \dot{x} &= k[y - F(x)] \\ \dot{y} &= \frac{1-x}{k} \end{aligned}$$

Period T is approximately the time required to travel along the two slow branches. On the slow branches, $y \approx F(x)$. $\frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (x^2 - 4)\frac{dx}{dt}$. Since $dy/dt = (1-x)/k, dx/dt = (1-x)/k(x^2-4)$. So $dt \approx \frac{k(x^2-4)}{1-x}dx$.

$$T = 2\int_4^2 \frac{k(x^2-4)}{1-x}dx = k(16 - 6ln3).$$

5. Strogatz: Problem 7.5.5
   Answer:

Let $F(x) = \int (|x| - 1)dx = x|x|/2 - x, y = \frac{-x}{\mu}$. Then

$$\begin{aligned} \dot{x} &= \mu[y - F(x)] \\ \dot{y} &= -\frac{x}{\mu} \end{aligned}$$

Period T is approximately the time required to travel along the two slow branches. On the slow branches, $y \approx F(x)$. $\frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (|x|-1)\frac{dx}{dt}$. Since $dy/dt = -x/\mu, dx/dt = -x/\mu(|x| - 1)$. So $dt \approx -\frac{\mu(|x|-1)}{x}dx$.

$$T = 2\int_{1+\sqrt{2}}^{1} -\frac{\mu(|x| - 1)}{x} = \mu(2\sqrt{2} - 2ln(1 + \sqrt{2})).$$