Dynamical Systems (550.391)
Homework 6 (Due: Thursday, November 3, 2005)

General Directions: You must show all work and document any assumptions to receive full credit on a problem. All problems are to be done by hand unless otherwise stated.

1. Recall that we have the following relationships between Cartesian coordinates (x and y) and polar coordinates (r and θ):

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

In class, we used these relations to show that

$$\tan\theta = \frac{y}{x},$$

and

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

(a) Show that

$$y\frac{dx}{dt} - x\frac{dy}{dt} = -r^2\frac{d\theta}{dt}.$$

(b) Convert the system below from Cartesian coordinates to polar coordinates. (That is, develop expressions for dr/dt and $d\theta/dt$.)

$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \qquad (1)$$

$$\frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \qquad (2)$$

(c) Determine the type and stability of any fixed points of this system.

(d) Determine the stability of any limit cycles of this system.

2. Use an appropriate theorem or criterion to show that the given system has no periodic solutions (other than constant solutions).

$$\frac{dx}{dt} = x + y + x^3 - y^2, \qquad (3)$$

$$\frac{dy}{dt} = -x + 2y + x^2y + y^3/3. \qquad (4)$$

3. Consider the system of equations

$$\frac{dx}{dt} = \mu x + y - x(x^2+y^2), \qquad (5)$$

$$\frac{dy}{dt} = -x + \mu y - y(x^2+y^2), \qquad (6)$$

where μ is a parameter.

(a) Show that the origin is the only critical point. (Hint: It may help to switch to polar coordinates.)

(b) Determine the type and stability of the critical point at the origin. How does this classification depend on μ?

(c) Show that when $\mu > 0$, there is a periodic solution. What is the equation for this limit cycle? Is it stable, unstable, or semi-stable?

4. There are certain chemical reactions in which the constituent concentrations oscillate periodically over time. The system

$$\frac{dx}{dt} = 1 - (b+1)x + \frac{x^2 y}{4}, \quad (7)$$

$$\frac{dy}{dt} = bx - \frac{x^2 y}{4}, \quad (8)$$

is a special case of a model, known as the *Brusselator*, of this kind of reaction. Assume that b is a positive parameter, and consider solutions in the first quadrant of the x, y-plane.

(a) Show that the only critical point is (1,4b).

(b) Determine the type and stability of the critical point. How does this classification depend on b?

(c) As b increases through a certain value b_0, the critical point changes its stability. What is b_0?

(d) Use a computer to plot trajectories in the phase plane for a value of b slightly less than b_0. Plot trajectories in the phase plane for a value of b slightly greater than b_0.

(e) Plot trajectories in the phase plane for four values of b greater than b_0 (not including the value you used in part (c)) and observe how the limit cycle deforms as b increases.

5. Strogatz: Problem 7.3.1