

Dynamical Systems(550.391)
Homework 5 (Due Thursday, October 27, 2005)

General Directions: You must show all work and document any assumptions to receive full credit on a problem. Feel free to use MATLAB or any other computer system to find the fixed points, eigenvalues, and, if necessary, eigenvectors for your analysis. (Refer to the Software Usage Guidelines on the course website to determine what computer output you need to submit.)

1. Strogatz: Problem 6.3.10

Answer:

(a) The Jacobian is $J = \begin{pmatrix} y & x \\ 2x & -1 \end{pmatrix}$, $J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

$|J(0,0)| = 0$, so linearization predicts that $(0,0)$ is non-isolated.

(b) $\dot{x} = \dot{y} = 0 \Rightarrow xy = 0, y = x^2 \Rightarrow x = y = 0$. Origin is the only fixed point, so it is isolated.

(c) Origin is not repelling, attracting or a saddle. It does not belong to any common type of fixed points.

(d) Phase portrait of this system is Figure 1.

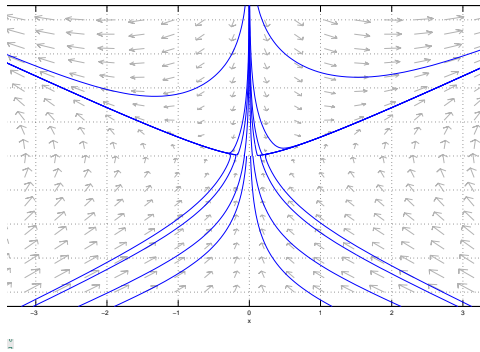


Figure 1

2. Strogatz: Problem 6.3.16

Answer:

(a) When $a = 0$, the phase portrait of this system is Figure 2.

(b) When $a > 0$, the phase portrait of this system is Figure 3.

When $a < 0$, the phase portrait of this system is Figure 4.

3. Strogatz: Problem 6.5.12

Answer:

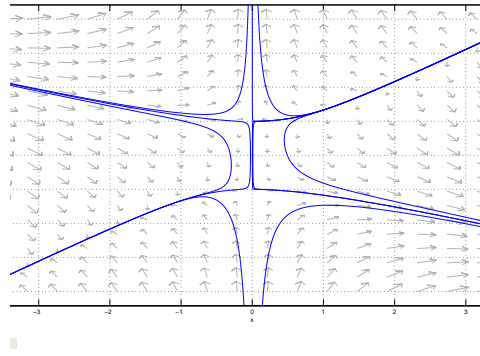


Figure 2

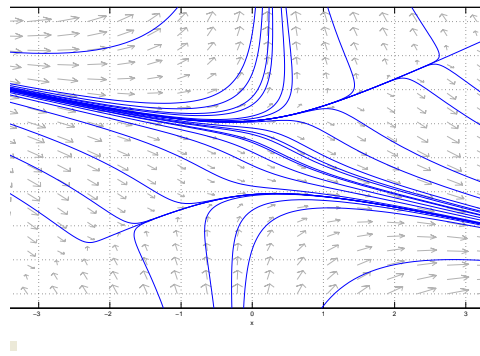


Figure 3

(a) $\frac{dE}{dt} = 2x\dot{x} + 2y\dot{y} = 2x \cdot (xy) + 2y \cdot (-x^2) = 0$, so E is conservative.

(b) At origin, $\dot{x} = \dot{y} = 0$, so origin is a fixed point.

$\forall k \in \mathbb{R}, (0, k)$ are fixed points. When $k \rightarrow 0$, $(0, k)$ approaches origin, so origin is not isolated.

(c) The phase portrait of this system is Figure 5.

4. Strogatz: Problem 6.6.6

Answer:

(a) The nullclines of this system is Figure 6.

(b) There are nine cases:

(a) $x < -y^2$ AND $(0 < y < 1$ OR $y < -1)$: $\dot{x} > 0, \dot{y} > 0$.

(b) $x = -y^2$ AND $(0 < y < 1$ OR $y < -1)$: $\dot{x} > 0, \dot{y} = 0$.

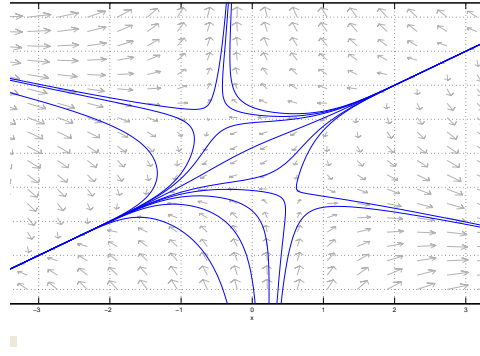


Figure 4

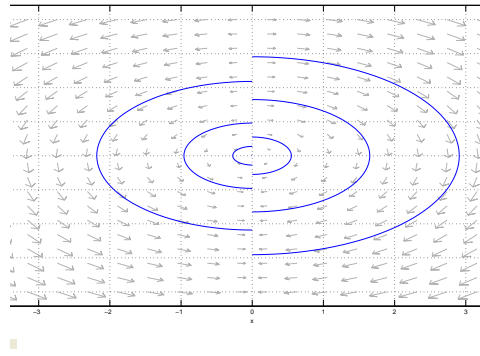


Figure 5

(c) $x > -y^2$ AND $(0 < y < 1$ OR $y < -1)$: $\dot{x} > 0, \dot{y} < 0$.

(d) $x < -y^2$ AND $(y = 0$ OR $y = 1$ OR $y = -1)$: $\dot{x} = 0, \dot{y} > 0$.

(e) $x = -y^2$ AND $(y = 0$ OR $y = 1$ OR $y = -1)$: $\dot{x} = 0, \dot{y} = 0$.

(f) $x > -y^2$ AND $(y = 0$ OR $y = 1$ OR $y = -1)$: $\dot{x} = 0, \dot{y} < 0$.

(g) $x < -y^2$ AND $(-1 < y < 0$ OR $y > 1)$: $\dot{x} > 0, \dot{y} > 0$.

(h) $x = -y^2$ AND $(-1 < y < 0$ OR $y > 1)$: $\dot{x} = 0, \dot{y} > 0$.

(i) $x > -y^2$ AND $(-1 < y < 0$ OR $y > 1)$: $\dot{x} < 0, \dot{y} > 0$.

(c) The Jacobian is $J = \begin{pmatrix} 0 & 1 - 3y^2 \\ -1 & -2y \end{pmatrix}$, $J(-1, +1) = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}$, $J(-1, -1) =$

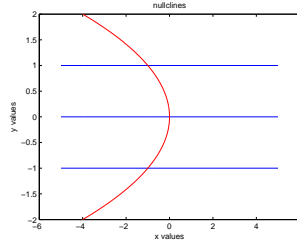


Figure 6

$$\begin{pmatrix} 0 & -2 \\ -1 & +2 \end{pmatrix}.$$

The characteristic equation of $(-1, +1)$ is $\lambda^2 + 2\lambda - 2 = 0$. Its eigenvalues and corresponding eigenvectors are $-1 + \sqrt{3} : (1, \frac{1-\sqrt{3}}{2})^T$; $-1 - \sqrt{3} : (1, \frac{1+\sqrt{3}}{2})^T$.

The characteristic equation of $(-1, -1)$ is $\lambda^2 - 2\lambda - 2 = 0$. Its eigenvalues and corresponding eigenvectors are $1 + \sqrt{3} : (1 - \sqrt{3}, 1)^T$; $1 - \sqrt{3} : (1 + \sqrt{3}, 1)^T$.

- (d) For $(-1, -1)$, the eigenvector corresponding to the unstable manifold is $(1 - \sqrt{3}, 1)^T$. At first, $\dot{x} < 0, \dot{y} > 0$. So if (x, y) move upward and to the left from $(-1, -1)$, $\dot{x} < 0, \dot{y} > 0$ remains until (x, y) crosses the x-axis (i.e. $y = 0$). Since x moves from -1 and $\dot{x} < 0$, the unstable manifold will intersect the negative x-axis. And since the system is reversible, it will have a twin trajectory that moves from negative x-axis to $(-1, 1)$, which is another fixed point, so a heteroclinic trajectory exists.
- (e) For $(-1, +1)$, the eigenvector corresponding to the unstable manifold is $(1, \frac{1-\sqrt{3}}{2})^T$. At first, $\dot{x} > 0, \dot{y} < 0$. So if (x, y) move downward and to the right from $(-1, +1)$, $\dot{x} > 0, \dot{y} < 0$ remains until (x, y) crosses the x-axis (i.e. $y = 0$). Since x moves from -1 and $\dot{x} > 0$, the unstable manifold will intersect the x-axis at a point on the right of -1 . And since the system is reversible, it will have a twin trajectory that moves from negative x-axis to $(-1, -1)$, which is another fixed point, so another heteroclinic trajectory exists.

The phase portrait of this system is Figure 7.

5. Strogatz: Problem 6.7.1

Answer:

Let $y = \dot{\theta}$, then

$$\begin{aligned} \dot{\theta} &= y \\ \dot{y} &= -\sin \theta - by \end{aligned}$$

$$\dot{\theta} = \dot{y} = 0 \Rightarrow y = 0, \sin \theta = 0 \Rightarrow (\theta, y) = (k\pi, 0), k \in \mathbb{Z}.$$

Since the flow is on the cylinder, we only need to consider two cases: $k=0$ and $k=1$.

$$\text{The Jacobian is } J = \begin{pmatrix} 0 & 1 \\ -\cos \theta & -b \end{pmatrix}.$$

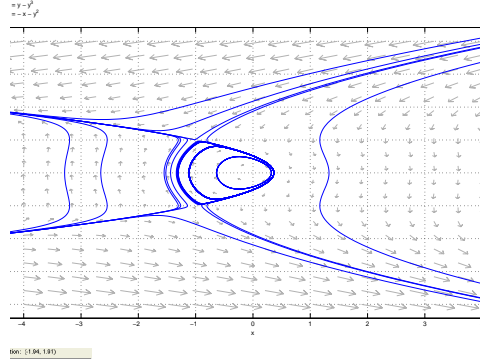


Figure 7

$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}$. $\tau = -b, \Delta = 1$. So when $b > 2$, it's a stable node (overdamped case); when $b = 2$, it's a stable degenerate node; when $0 < b < 2$, it's a stable spiral (underdamped case).

$J(\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix}$. $\tau = -b, \Delta = -1$. So it's a saddle point (unstable).

Underdamped case: see figure 8.

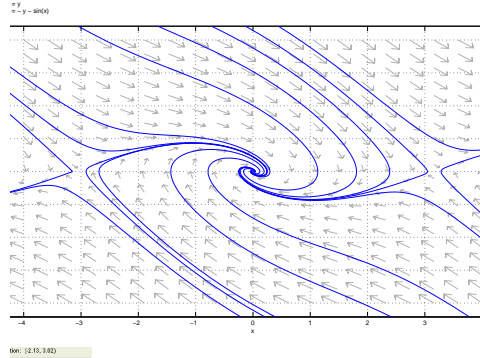


Figure 8

Overdamped case: see figure 9.

6. Strogatz: Problem 6.7.3

Answer: Let $y = \dot{\theta}$, then

$$\begin{aligned} \dot{\theta} &= y \\ \dot{y} &= -\sin \theta - (1 + a \cos \theta)y \end{aligned}$$

$\dot{\theta} = \dot{y} = 0 \Rightarrow y = 0, \sin \theta = 0 \Rightarrow (\theta, y) = (k\pi, 0), k \in \mathbb{Z}$.

Since the flow is on the cylinder, we only need to consider two cases: $k=0$ and $k=1$.

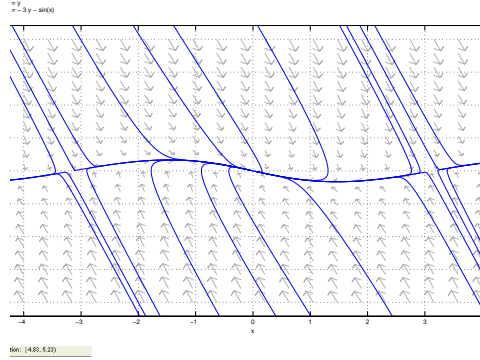


Figure 9

The Jacobian is $J = \begin{pmatrix} 0 & 1 \\ -\cos \theta + ay \sin \theta & -(1 + a \cos \theta) \end{pmatrix}$.

$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -a - 1 \end{pmatrix}$. $\tau = -(a + 1)$, $\Delta = 1$. So when $a > 1$, it's a stable node (overdamped case); when $a = 1$, it's a stable degenerate node; when $0 \leq a < 1$, it's a stable spiral (overdamped case).

$J(\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & a - 1 \end{pmatrix}$. $\tau = a - 1$, $\Delta = -1$. So it's a saddle point (unstable).

Underdamped case: see figure 10.

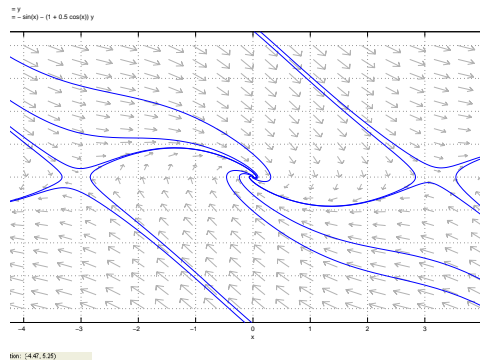


Figure 10

Overdamped case: see figure 11.

