Dynamical Systems(550.391)
Homework 4 (Due Thursday, October 20, 2005)

General Directions: You must show all work and document any assumptions to receive full credit on a problem. Feel free to use MATLAB or any other computer system to find the fixed points, eigenvalues, and, if necessary, eigenvectors for your analysis. (Refer to the Software Usage Guidelines on the course website to determine what computer output you need to submit.)

1. Peaceful Coexistence. Consider the almost linear system

$$\frac{dx}{dt} = 60x - 3x^2 - 4xy \quad (1)$$

$$\frac{dy}{dt} = 42y - 3y^2 - 2xy \quad (2)$$

(a) Briefly describe this system as an ecological model, using the relationship between the competition and inhibition effects to explain why it is called "peaceful coexistence."

(b) What are the fixed points of this system?

(c) Discuss the type and stability of each fixed point found in part (b).

(d) Sketch or use a computer to draw a phase portrait for this system.

Answer:

(a) The terms $-3x^2$ and $-3y^2$ represent inhibition. The terms $-4xy$ and $-2xy$ represent competition. $2 \times 2 < 3 \times 3$, so the effect of competition is weaker than the effect of inhibition. Thus it's called "peaceful coexistence".

(b) $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \Rightarrow (x, y) = (0,0), (0,14), (20,0), (12,6)$.

(c) The Jacobian is $J = \begin{pmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{pmatrix}$

So $(0,0)$ is a nodal source, $(0,14)$ is a saddle point(unstable), $(20,0)$ is a saddle point(unstable), $(12,6)$ is a nodal sink.

(d) Phase portrait of this system is Figure 1.

2. Predator-Prey. Consider the almost linear system

$$\frac{dx}{dt} = 5x - x^2 - xy \quad (3)$$

$$\frac{dy}{dt} = -2y + xy \quad (4)$$

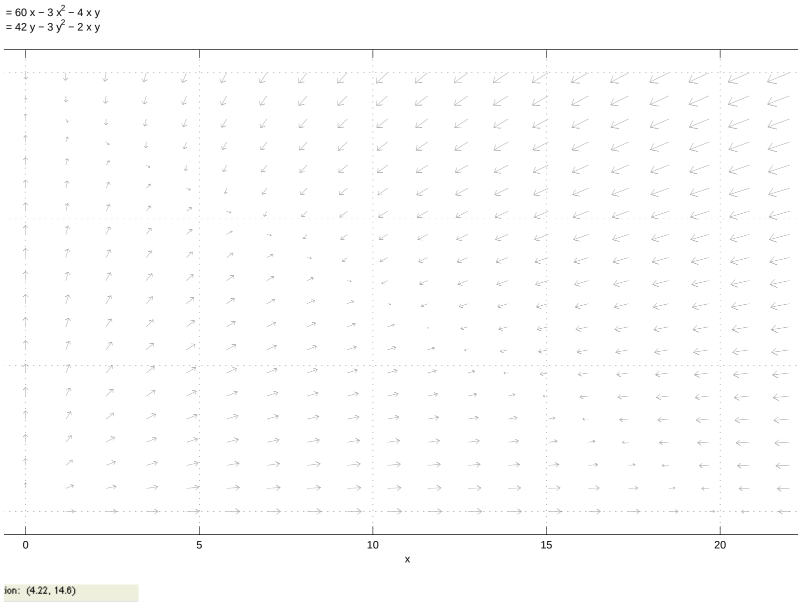

Figure 1

(a) Briefly describe this system as an ecological model, using the relationship between the interaction coefficients to explain why it could be called a "predation model."

(b) What are the fixed points of this system?

(c) Discuss the type and stability of each fixed point found in part (b).

(d) Sketch or use a computer to draw a phase portrait for this system.

Answer:

(a) It is called predation models because they look like predator-prey (xy in one population's equation means interaction leads to growth; $-xy$ in the other means interaction leads to death). x represents the predator; y represents the prey.

(b) $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \Rightarrow (x, y) = (0, 0), (5, 0), (2, 3)$.

(c) The Jacobian is $J = \begin{pmatrix} 5 - 2x - y & -x \\ y & -2 + x \end{pmatrix}$

So $(0, 0)$ is a saddle point(unstable), $(5, 0)$ is a saddle point(unstable), $(2, 3)$ is a spiral sink.

(d) Phase portrait of this system is Figure 2.

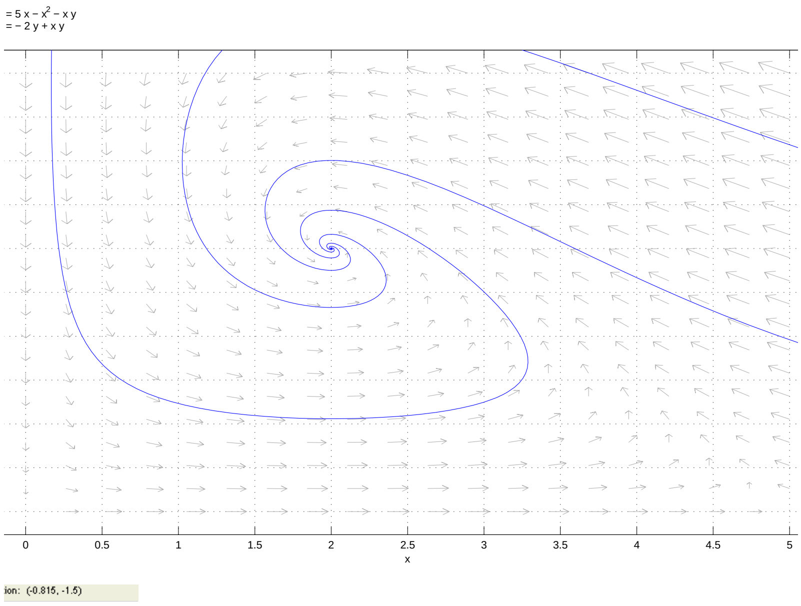

Figure 2

3. Unsophisticated Predator-Prey. Consider the almost linear system

$$\frac{dx}{dt} = x^2 - 2x - xy \tag{5}$$

$$\frac{dy}{dt} = y^2 - 4y + xy \tag{6}$$

(a) Briefly describe this system as an ecological model, using the relationship between the interaction coefficients to explain why it could be called a "predation model." Also explain why the populations would be called "unsophisticated."

(b) What are the fixed points of this system?

(c) Discuss the type and stability of each fixed point found in part (b).

(d) Sketch or use a computer to draw a phase portrait for this system.

Answer:

(a) It is called predation models because they look like predator-prey (xy in one population's equation means interaction leads to growth; $-xy$ in the other means interaction leads to death). x represents the predator; y represents the prey.

It is called unsophisticated because the underlying model (without interaction terms) is not logistic. It has $ax^2 - bx$ rather than $-ax^2 + bx$. Here mating relies on chance encounters so the population faces either doomsday (infinite population) or extinction.

(b) $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \Rightarrow (x, y) = (0, 0), (0, 4), (2, 0), (3, 1).$

(c) The Jacobian is $J = \begin{pmatrix} 2x - 2 - y & -x \\ y & 2y - 4 + x \end{pmatrix}$

So $(0, 0)$ is a nodal sink, $(0, 4)$ is a saddle point(unstable), $(2, 0)$ is saddle point(unstable), $(3, 1)$ is a spiral source.

(d) Phase portrait of this system is Figure 3.

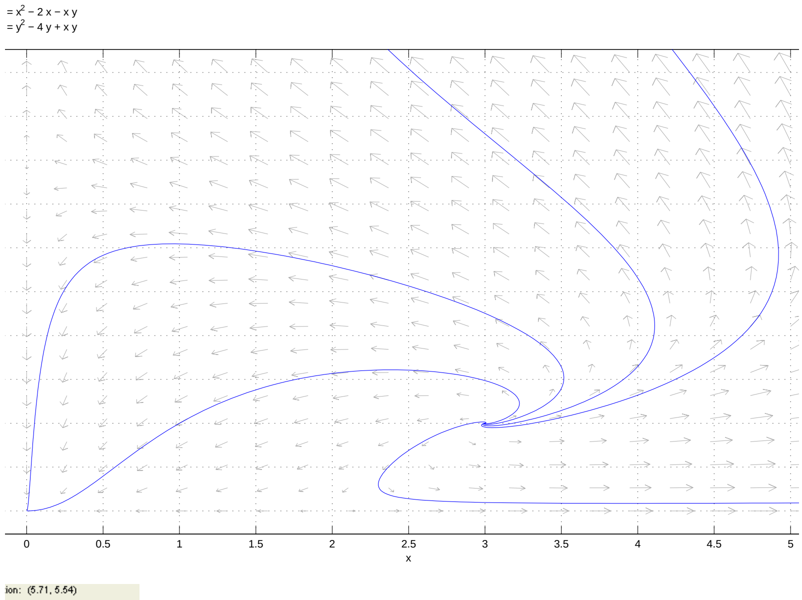

Figure 3

4. In the case of a two-dimensional system that is NOT almost linear, the trajectories near an isolated critical point can exhibit a considerably more complicated structure than those near nodes, centers, saddle points, and spiral points. For example, consider the system

$$\frac{dx}{dt} = x(x^3 - 2y^3) \tag{7}$$

$$\frac{dy}{dt} = y(2x^3 - y^3) \tag{8}$$

(a) Explain why (0,0) is NOT an isolated critical point of the associated linearized system.

(b) The trajectories of this system may be determined by solving the first-order homogenous equation

$$\frac{dy}{dx} = \frac{y(2x^3 - y^3)}{x(x^3 - 2y^3)}.$$

Solve this equation by using the substitution $y = vx$ (hence $dy = vdx + xdv$) and separating the resulting variables.

(c) The trajectories of this system are *folia of Descartes*. Sketch or use a computer to draw some representative trajectories for this system.

Answer:

(a) The Jacobian is $J = \begin{pmatrix} 4x^3 - 2y^3 & -6xy^2 \\ 6x^2y & 2x^3 - 4y^3 \end{pmatrix}$, $J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $|J(0,0)| = 0$, so (0,0) is non-isolated.

(b) The substitution $y = vx$ in the homogeneous first-order equation

$$\frac{dy}{dx} = \frac{y(2x^3 - y^3)}{x(x^3 - 2y^3)}$$

yields

$$x\frac{dv}{dx} = -\frac{v^4 + v}{2v^3 - 1}.$$

Separating the variables and integrating by partial fractions, we get

$$\int (-\frac{1}{v} + \frac{1}{v+1} + \frac{2v-1}{v^2 - v + 1})dv = -\int \frac{dx}{x}$$

$$ln((v+1)(v^2 - v + 1)) = lnv - lnx + lnC$$

$$(v+1)(v^2 - v + 1) = \frac{Cv}{x}$$

$$v^3 + 1 = \frac{Cv}{x}.$$

Finally, the replacement $v = y/x$ yields $x^3 + y^3 = Cxy$.

(c) Phase portrait of this system is Figure 4.

5. Interesting and complicated phase portraits often result from simple nonlinear perturbations of linear systems. In this problem, we consider the almost linear system

$$\frac{dx}{dt} = -y\cos(x + y - 1) \quad (9)$$

$$\frac{dy}{dt} = x\cos(x - y + 1) \quad (10)$$

(a) Use MATLAB or some other computer software to draw the phase portrait for system in the range $-3 \leq x, y \leq 3$.

(b) Determine the fixed points in this region of the phase plane.

(c) Based on your phase portrait, discuss the "type" of each fixed point identified in part (b).

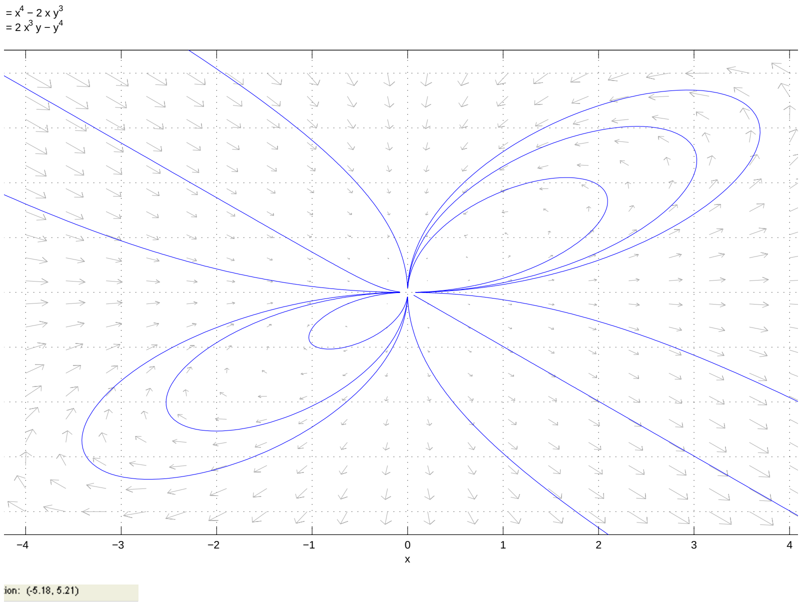

Figure 4

Answer:

(a) Phase portrait of this system is Figure 5.

(b) $\dot{x} = 0 \Leftrightarrow y = 0$ or $x + y - 1 = m \cdot \frac{\pi}{2}$; $\dot{y} = 0 \Leftrightarrow x = 0$ or $x - y + 1 = n \cdot \frac{\pi}{2}$, where m, n are odd integers.
So, the fixed points are $(0,0), (0, \frac{m\pi}{2} + 1), (\frac{n\pi}{2} - 1, 0), (\frac{(m+n)\pi}{4}, \frac{(m-n)\pi}{4} + 1)$, where m, n are odd integers.
In the range $-3 \leq x, y \leq 3$, fixed points are:
$(0,0), (0, 1 - \frac{\pi}{2}), (0, 1 + \frac{\pi}{2}), (\frac{\pi}{2} - 1, 0), (-\frac{\pi}{2} - 1, 0)$,
$(\frac{\pi}{2}, 1), (-\frac{\pi}{2}, 1), (\frac{\pi}{2}, 1 - \pi), (-\frac{\pi}{2}, 1 - \pi)$.

(c) Type of fixed points:
$(0,0)$: spiral
$(0, 1 - \frac{\pi}{2})$: fixed points of undetermined character
$(0, 1 + \frac{\pi}{2})$: fixed points of undetermined character
$(\frac{\pi}{2} - 1, 0)$: saddle point
$(-\frac{\pi}{2} - 1, 0)$: spiral
$(\frac{\pi}{2}, 1)$: spiral
$(-\frac{\pi}{2}, 1)$: saddle point
$(\frac{\pi}{2}, 1 - \pi)$: saddle point
$(-\frac{\pi}{2}, 1 - \pi)$: spiral

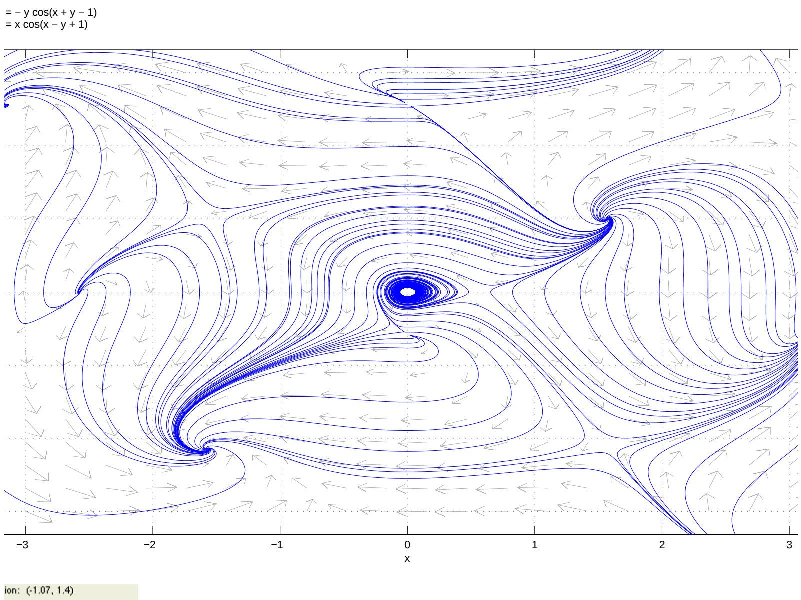

Figure 5