Dynamical Systems(550.391)
Homework 3 (Due: Thursday, October 6, 2005)

General Directions: You must show all work and document any assumptions to receive full credit on a problem. Feel free to use a computer to create any required graphics or to determine the location of specific fixed points.

1. Strogatz: Problem 3.4.16; (See Section 2.7 for a discussion of Potentials)
   Answer:

   (a) $-dV/dx = r - x^2 \Rightarrow V = \frac{x^3}{3} - rx$.
   Bifurcation occurs at $r = 0$, graphs are as figure 1:

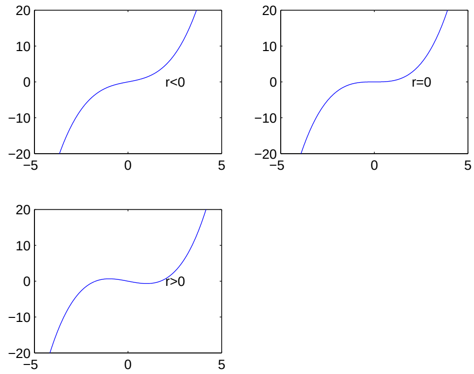

Figure 1

   (b) $-dV/dx = rx - x^2 \Rightarrow V = \frac{x^3}{2} - \frac{r}{2}x^2$.
   Bifurcation occurs at $r = 0$, graphs are as figure 2:

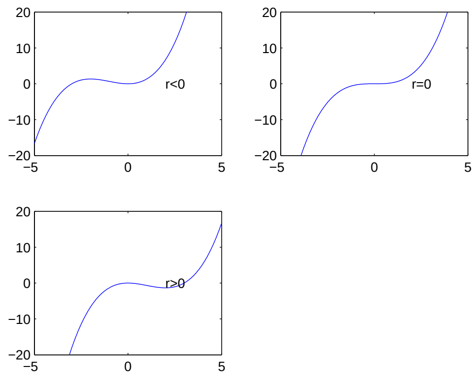

Figure 2

(c) $-dV/dx = rx + x^3 - x^5 \Rightarrow V = \frac{x^6}{6} - \frac{x^4}{4} - \frac{r}{2}x^2$. Bifurcation occurs at $r = 0$, graphs are as figure 3:

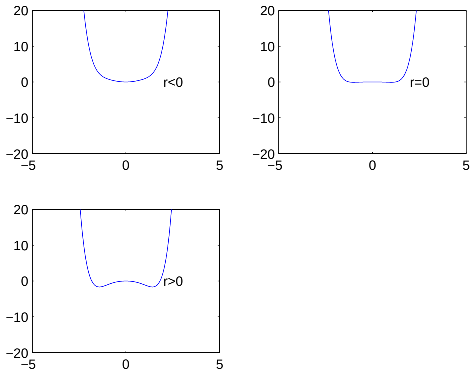

Figure 3

2. Strogatz: Problem 3.5.8
Answer:
$du/dt = au + bu^3 - cu^5$.
Let $x = u/U, \tau = t/T$, then $\frac{Udx}{Td\tau} = aUx + bU^3x^3 - cU^5x^5 \Rightarrow dx/d\tau = aTx + bU^2Tx^3 - cU^4Tx^5$.
Let $bU^2T = 1, cU^4 = 1$, then $U = \sqrt{b/c}, T = c/b^2$. $r = aT = \frac{ac}{b^2}$.

3. Strogatz: Problem 3.6.2(a), (b)
Answer:

(a) We could use graphs or stability analysis to classify the stability of the fixed points.
When $h = 0$, it is a standard transcritical bifurcation.
When $h > 0$, three are 2 saddle-node bifucation.
When $h < 0$, no bifurcation occurs.
The bifurcation diagram is figure 4.

(b) The stability diagram is figure 5.

Boundary cases: On the curve $r^2 = -4h$, saddle mode bifurcation occurs.
On the line $r = 0$, transcritical bifurcation occurs.

4. Strogatz: Problem 3.6.3
Answer:

(a) We could use graphs or stability analysis to classify the stability of the fixed points.
When $a = 0$, it is a standard supercritical pitchfork bifurcation.
When $a > 0$, a saddle-node bifurcation occurs at $r = -\frac{a^2}{4}$ and a transcritical bifurcation occurs at $r = 0$.

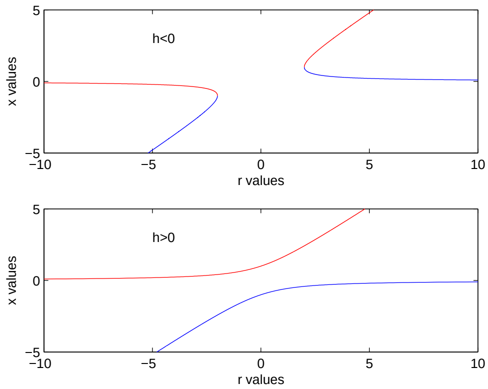

Figure 4

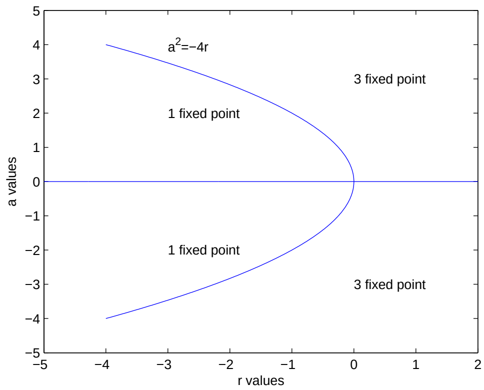

Figure 5

When $a < 0$, a saddle-node bifurcation occurs at $r = -\frac{a^2}{4}$ and a transcritical bifurcation occurs at $r = 0$.
The bifurcation diagram is figure 6.

(b) The stability diagram is figure 7.

Boundary cases: On the curve $a^2 = -4r$, saddle mode bifurcation occurs.
On the line $r = 0$, transcritical bifurcation occurs.

5. Strogatz: Problem 3.6.4
Answer:
The standard form of a saddle-node bifurcation is $\dot{x} = r + x^2$. When imperfection occurs, $\dot{x} = h + r + x^2$.
Let $r' = h + r$ (given an small imperfection h is fixed), then $\dot{x} = r' + x^2$, this is a standard form, its a typical saddle-node bifurcation. It means the curve on the bifurcation diagram in (r', x) plane is standard. Then we use $r = r' - h$ to return to the imperfection case. It means the curve on the bifurcation diagram in (r, x) plane

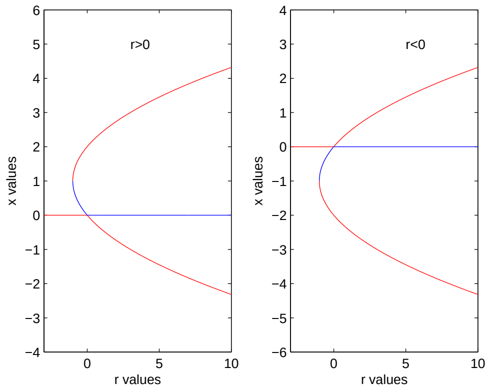

Figure 6

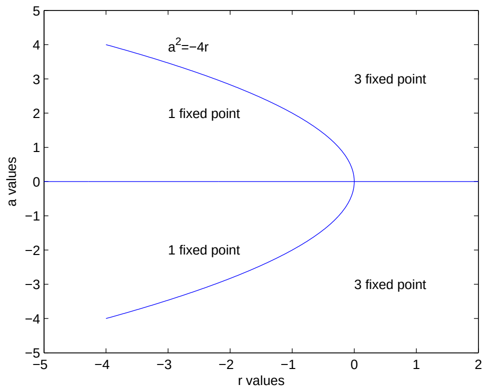

Figure 7

will shift to the right when $h < 0$; or shift to the left when $h > 0$, but the shape remains the same.

6. Strogatz: Problem 3.7.2(a)
   Answer:
   Graphs are as figure 8
   $\lim_{x\to 1} r(x) = 0.5$, $\lim_{x\to\infty} r(x) = 0$.
   $\lim_{x\to 1^-} k(x) = -\infty$, $\lim_{x\to 1^+} k(x) = +\infty$, $\lim_{x\to\infty} k(x) = \infty$.

7. Strogatz: Problem 3.7.4(b)–(f)
   Answer:

   (b) $\dot{N} = rN(1 - \frac{N}{K}) - H\frac{N}{A+N}$.
   Let $x = N/K, a = A/K, h = H/rk, \tau = rt$, then the system could be rewritten in dimensionless form as $\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x}$.
   From the biological interpretation, we know a,x and h are nonnegative numbers.

   (c) According to the biological interprecation, $x \geq 0$.

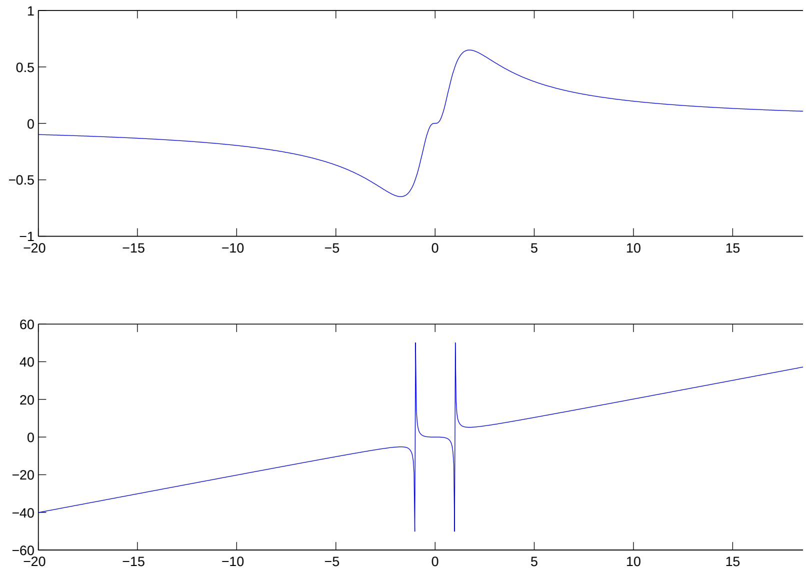

Figure 8

$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} = -\frac{x}{a+x}[x^2 + (a-1)x + h - a]$. $x = 0$ is always a fixed point. Other fixed points are nonnegative roots of $x^2 + (a-1)x + h - a$.

If $a < 1$;
when $h > \frac{(a+1)^2}{4}$, there is one fixed point.
when $h = \frac{(a+1)^2}{4}$ or $h \geq a$, there are two fixed points.
when $a < h < \frac{(a+1)^2}{4}$, there are three fixed points.

If $a = 1$;
when $h \geq 1$, there is one fixed point.
when $h < 1$, there are two fixed points.

If $a > 1$;
when $h \leq a$, there is one fixed point.
when $h > a$, there are two fixed points.

(Note: only nonnegative fixed points counts.)

(d) When x is near 0, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$.
$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} = x - x^2 - \frac{h}{a}x\frac{1}{1+\frac{x}{a}} = x - x^2 - \frac{h}{a}x(1 - \frac{x}{a}) + O(x^3)$
$= (1 - \frac{h}{a})x + (1 - \frac{h}{a^2})x^2 + O(x^3)$.
So a transcritical bifurcation occurs when $1 - \frac{h}{a} = 0$, which is equivalent to $h = a$.

(e) As stated in Part c, a bifurcation occurs at $h = \frac{1}{4}(a+1)^2$. It is a saddle mode bifurcation.

(f) The stability diagram is figure 9.

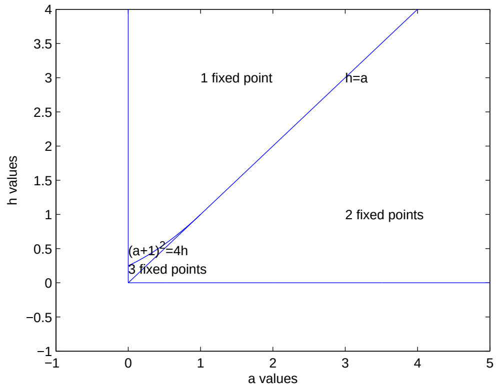

Figure 9

To see boundary cases, please refer to Part c.