Dynamical Systems (550.391)
Homework Solution 2

General Directions: You must show all work and document any assumptions to receive full credit on a problem.

Part I: Linear Stability Analysis

1. Strogatz,Problem 2.4.2
   Answer:
   $dx/dt = f(x) = x(1-x)(2-x) = x^3 - 3x^2 + 2x$. $f(x) = 0 \Rightarrow x = 0, 1, 2$.
   $f'(x) = 3x^2 - 6x + 2$. $f'(0) > 0$, so 0 is an unstable fixed point. $f'(1) < 0$, so 1 is a stable fixed point. $f'(2) > 0$, so 2 is an unstable fixed point.

2. Strogatz, Problem 2.4.4
   Answer:
   $dx/dt = f(x) = x^2(6-x) = 6x^2 - x^3$. $f(x) = 0 \Rightarrow x = 0, 6$.
   $f'(x) = 12x - 3x^2$. $f'(6) < 0$, so 6 is a stable fixed point. $f'(0) = 0$, so we can't classify the stability of 0 by stability analysis. When $x < 0, f(x) > 0$, and when $0 < x < 6, f(x) > 0$. So 0 is a half-stable point.

Part II: Existence and Uniqueness

1. Strogatz, Problem 2.5.1
   Answer:

   (a) $dx/dt = f(x) = -x^c$. If $x = 0$ is a fixed point, then $f(0) = 0$. So $-0^c = 0 \Rightarrow c > 0$. When $c > 0$ and $x > 0$, $-x^c < 0$. Since $x \geq 0$, 0 is stable. Therefore, if $c > 0$, 0 is a stable fixed point.

   (b) Suppose $c > 0$. $dx/dt = -x^c$.
   When $c \neq 1, dx/(x^c) = -dt \Rightarrow \frac{1}{1-c}x^{1-c} = a - t$ (a is a constant). So $t = a - \frac{1}{1-c}x^{1-c}$.
   When $c = 1$, $dx/x = -dt \Rightarrow ln(x) = a - t$
   If $c > 1$, $x^{1-c}|_{x=0}$ is infinite; if $c = 1$, $ln(x)|_{x=0}$ is infinite. So if $c \geq 1$, the particle can't reach the origin in a finite time.
   If $0 < c < 1$, the particle can reach the origin in a finite time. When $x = 1$, t=$a - \frac{1}{1-c}$; when $x = 1$, t=a. So it takes $a - (a - \frac{1}{1-c}) = \frac{1}{1-c}$ for the particle to travel from $x = 1$ to $x = 0$.

   Use the Existence and Uniqueness Theorem (see p.27 in the text) to discuss the existence and uniqueness of solutions to the following initial value problems (IVPs):

2. $dx/dt = f(x) = ln(1 + x^2) \quad x(0) = 0$
   Answer:
   $dx/dt = f(x) = ln(1 + x^2)$. $f'(x) = 2x/(1 + x^2)$. Both $f(x)$ and $f'(x)$ are continuous on a small neighbor of 0. So we could apply the existence and uniqueness theorem, e.g. the solution of this problem is unique.

   Notice: actually we could "guess" one solution: $x(t) = 0$ (a constant function). According to the existence and uniqueness theorem, this is the only solution.

3. $dx/dt = f(x) = -\sqrt{1 - x^2} \quad x(0) = 1$ on the interval $[0, \pi]$
   Answer:
   $dx/dt = f(x) = -\sqrt{1 - x^2}$. $f'(x) = x/\sqrt{1 - x^2}$. $f'(x)$ is not continuous at 1. So we could not apply the existence and uniqueness theorem, e.g. the solution of this problem may not be unique.

   Notice: actually we could find two solutions to this problem (maybe we could find more!): $x(t) = 1$ (a constant function) and $x(t) = \cos t$. As expected, the solution to this problem is not unique.

   (Note: For a discussion of slope fields, see example 2.8.1.)By hand, create the slope field for the following differential equations. Augment your graphic by sketches of representative solution curves.

4. $dx/dt = -\sqrt{1 - x^2}$
   Answer:
   See figure 1.

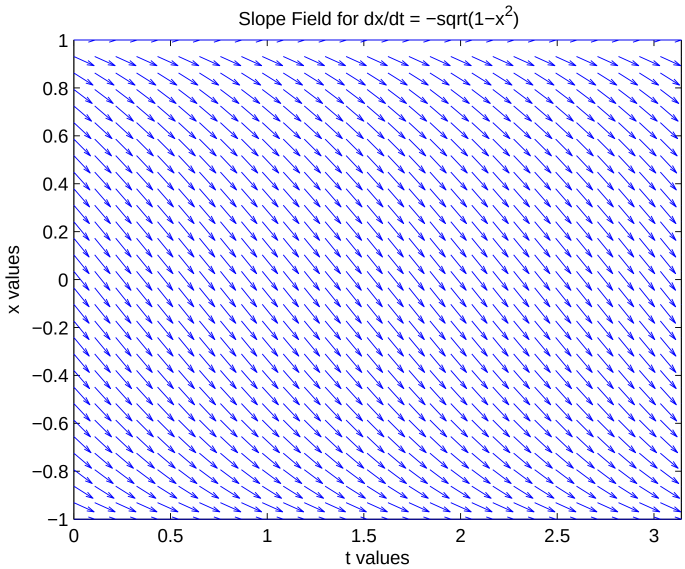

Figure 1

5. (See Section 2.1 for a discussion of this ODE.)
   $dx/dt = sinx$
   Answer:
   See figure 2.

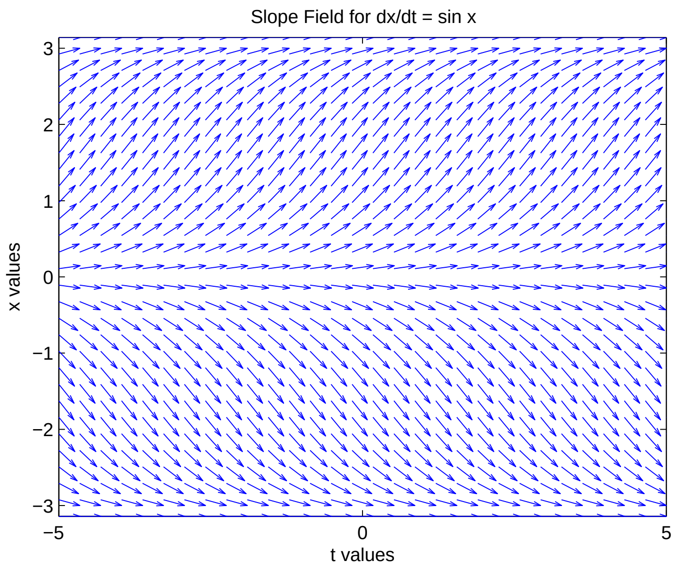

Figure 2

Part III: Saddle-Node Bifurcation

1. Strogatz, Problem 3.1.4
   Answer:
   $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$. Let $f(x) = r + \frac{1}{2}x$, $g(x) = \frac{x}{1+x}$. Bifurcation occurs when $f(x) = g(x)$, $f'(x) = g'(x)$. So $r = 1.5 \pm \sqrt{2}$.

   Figure 3 is the bifurcation diagram (red indicates stable fixed points):

2. Strogatz, Problem 3.1.5(a)
   Answer:
   Similar to the saddle-node bifurcation, bifurcation occurs when $r = 0$.

   Figure 4 is the bifurcation diagram (red indicates stable fixed points):

Part IV: Transcritical Bifurcation

1. Strogatz, Problem 3.2.2
   Answer:
   $\dot{x} = rx - ln(1 + x)$. Let $f(x) = rx$, $g(x) = ln(1 + x)$. Bifurcation occurs when

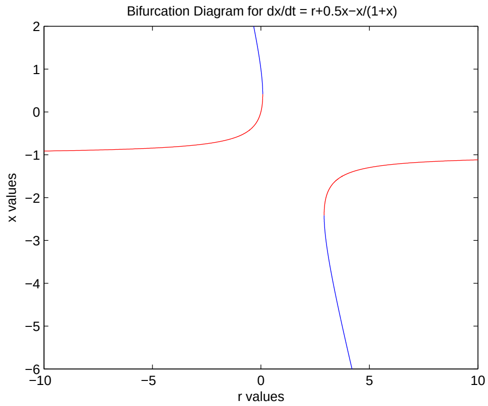

Figure 3

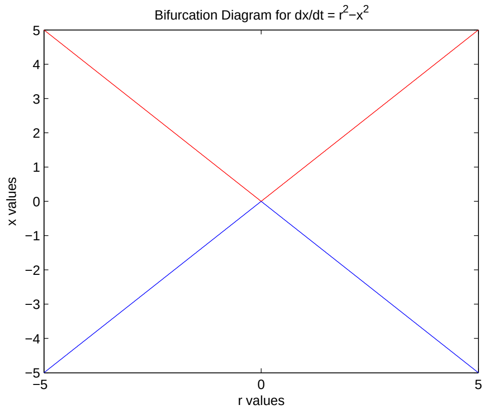

Figure 4

$f(x) = g(x)$, $f'(x) = g'(x)$. So $r = 0$.

Figure 5 is the bifurcation diagram (red indicates stable fixed points):

2. Strogatz, Problem 3.2.3
   Answer:
   $\dot{x} = x - rx(1-x)$. Let $f(x) = x$, $g(x) = rx(1-x)$. Bifurcation occurs when $f(x) = g(x)$, $f'(x) = g'(x)$. So $r = 1$.

   Figure 6 is the bifurcation diagram (red indicates stable fixed points):

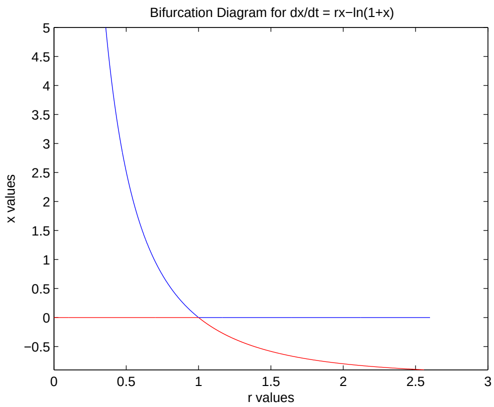

Figure 5

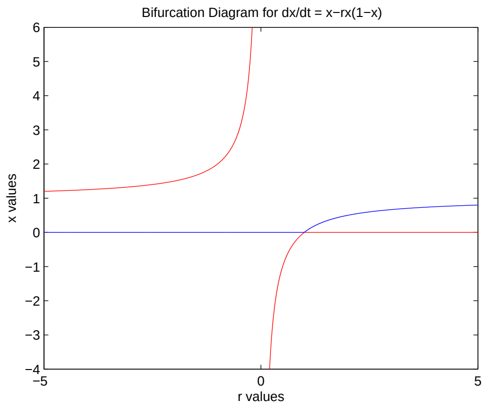

Figure 6