Dynamical Systems (550.391)
Homework 1 (Due Thursday, September 22, 2005)

General Directions: You must show all work and document any assumptions to receive full credit on a problem.

Part I: Population Models

1. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long will it take until all the fish are dead?

2. A tumor may be regarded as a population of multiplying cells. It is found empirically that the "birth rate" of the cells in a tumor decreases exponentially with time so that $\beta(t) = \beta_0 e^{-\alpha t}$ (where $\alpha, \beta_0 > 0$, constant). Hence

$$\frac{dP}{dt} = \beta_0 e^{-\alpha t} P, \qquad P(0) = P_0.$$

Solve this differential equation. What is $\lim_{t\to\infty} P(t)$?

Answer:

1. Suppose $\delta = a/\sqrt{P}$, then $dP/(Pdt) = (\beta - \delta) = -a/\sqrt{P}$. So $P = ((b - at)/2)^2$.
P(0)=900, P(6)=441, so a=3, b=60.
$P(t) = 0 \Rightarrow t = 20$. That means it will take 20 weeks until all the fish are dead.

2.

$$\frac{dP}{P} = \beta_0 e^{-\alpha t} dt$$

$$P(t) = C \cdot e^{-\frac{\beta_0}{\alpha} e^{-\alpha t}}$$

$$P(0) = P_0 \Rightarrow C = P_0 e^{\frac{\beta_0}{\alpha}}$$

$$P(t) = P_0 e^{\frac{\beta_0}{\alpha}(1 - e^{-\alpha t})}$$

So

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} C \cdot e^{-\frac{\beta_0}{\alpha} e^{-\alpha t}} = C = P_0 e^{\frac{\beta_0}{\alpha}}$$

Part II: Critical Points and the Phase Portrait

The problems below involve equations of the form $dx/dt = f(x)$. For each problem, sketch the graph of $f(x)$ versus x, determine the fixed (critical) points, classify each point as stable, unstable, or semistable, and draw the phase line.

1.

$$\frac{dx}{dt} = -(x-1)^2$$

2.

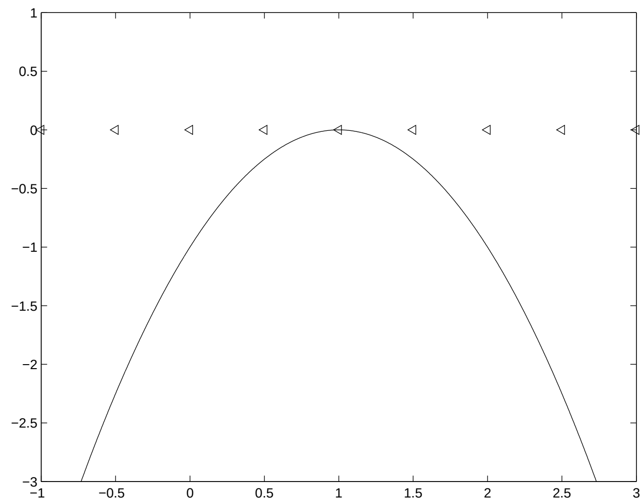
$$\frac{dx}{dt} = x^2(x^2-1)$$

3.

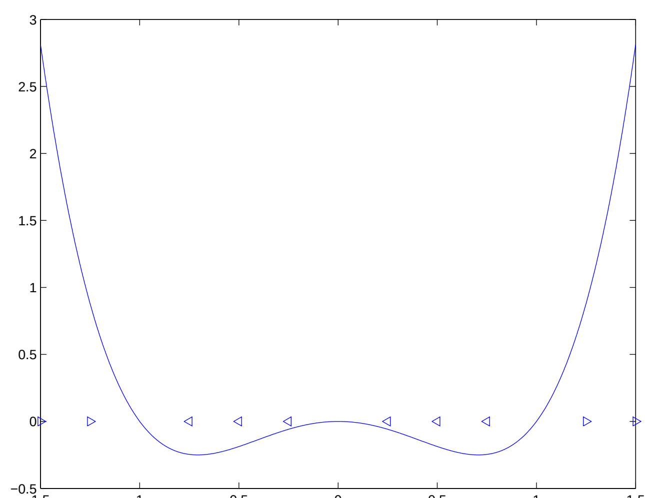
$$\frac{dx}{dt} = x(1-x^2)$$

Answer:

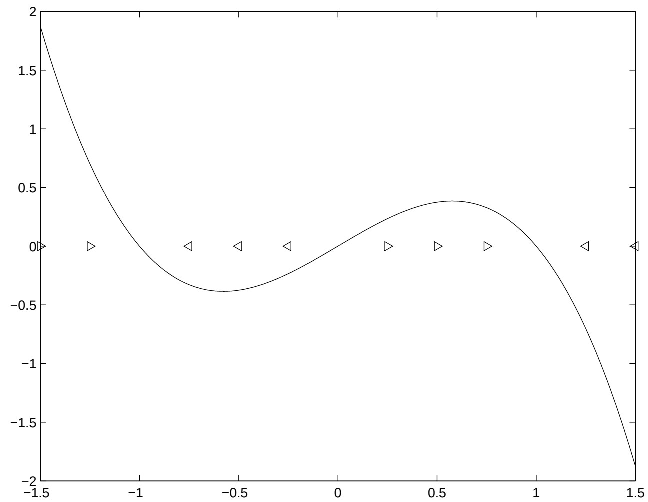
1. Critical point: x=1, semistable.



2. Critical points: x=-1, stable; x=0, semistable; x=1, unstable.



3. Critical points: x=-1, stable; x=0, unstable; x=1, stable.



Part III: Solving First-Order ODEs

For each of the functions presented in Part II, find a general solution. Sketch several graphs of particular solutions for each ODE in tx-plane. (Please solve the equation by hand. Feel free to use computer software to create your graphs.)

Hint: You may find it helpful to setup the following partial fractions problems:

$$\frac{1}{x^2(x^2-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$$

and

$$\frac{1}{x(1-x^2)} = \frac{a}{x} + \frac{b}{1-x} + \frac{c}{1+x}.$$

Answer:

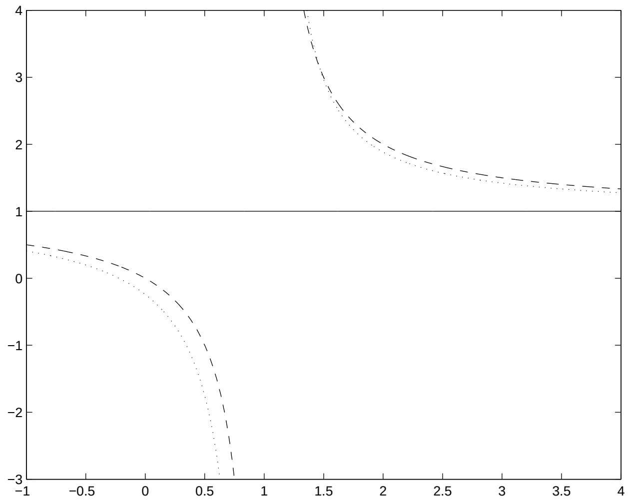
1.

$$\frac{dx}{-(x-1)^2} = dt$$

$$\int \frac{dx}{-(x-1)^2} = \int dt$$

$$\frac{1}{x-1} = t + C$$

(The figure below shows the solutions through (0,0), (0,1), and (0,-3).



2. Suppose

$$\frac{1}{x^2(x^2-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1},$$

Then

$$A = (x^2 \cdot \frac{1}{x^2(x^2-1)})'|_{x=0} = 0,$$

$$B = x^2 \cdot \frac{1}{x^2(x^2-1)}|_{x=0} = -1,$$

$$C = (x-1) \cdot \frac{1}{x^2(x^2-1)}|_{x=1} = \frac{1}{2},$$

$$D = (x+1) \cdot \frac{1}{x^2(x^2-1)}|_{x=-1} = -\frac{1}{2}.$$

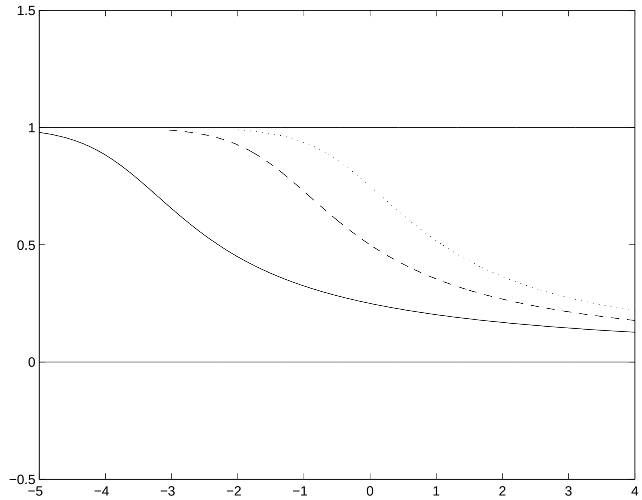
So

$$\frac{1}{x^2(x^2-1)} = \frac{-1}{x^2} + \frac{1}{2(x-1)} + \frac{-1}{2(x+1)},$$

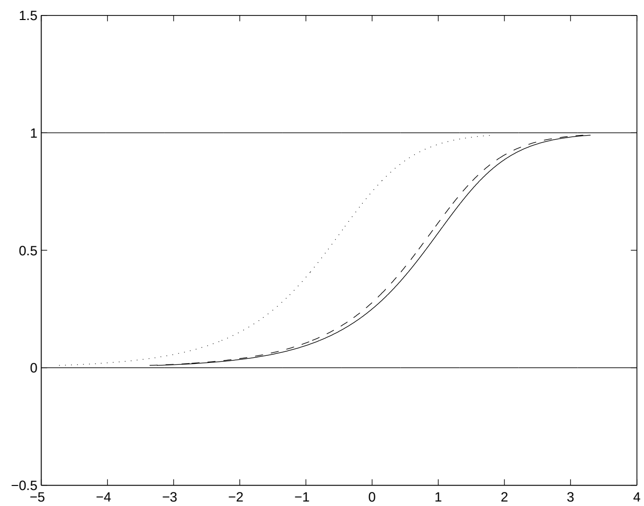
$$\frac{-dx}{x^2} + \frac{dx}{2(x-1)} + \frac{-dx}{2(x+1)} = dt,$$

$$\frac{1}{x} + \frac{1}{2}ln|x-1| - \frac{1}{2}ln|x+1| = t + C.$$

The figure below shows the solutions through (0,0), (0,0.25), (0,0.5), (0,0.75), and (0,1). (Question: What do solutions through (t_0, x_0) look like when $x_0 < 0$?)



3. Suppose
$$\frac{1}{x(1-x^2)} = \frac{a}{x} + \frac{b}{1-x} + \frac{c}{1+x}.$$
Then,
$$a = x \cdot \frac{1}{x(1-x^2)}|_{x=0} = 1,$$
$$b = (1-x) \cdot \frac{1}{x(1-x^2)}|_{x=1} = \frac{1}{2},$$
$$c = (1+x) \cdot \frac{1}{x(1-x^2)}|_{x=-1} = -\frac{1}{2}.$$
So
$$\frac{1}{x(1-x^2)} = \frac{1}{x} + \frac{1}{2(1-x)} + \frac{-1}{2(1+x)},$$
$$\frac{dx}{x} + \frac{dx}{2(1-x)} + \frac{-dx}{2(1+x)} = dt,$$
$$lnx - \frac{1}{2}ln|1-x| - \frac{1}{2}ln|1+x| = t + C$$
The figure below shows the solutions through (0,0), (0,0.25), (0,0.5), (0,0.75), and (0,1). (Question: What do solutions through (t_0, x_0) look like when $x_0 < 0$?)



Part IV: Bifurcations

1. Consider the differential equation $dx/dt = kx - x^3$.

   (a) If $k \leq 0$, show that there is only one critical point. What is that point? Discuss its stability.

   (b) If $k > 0$, show that there are three critical points. What are they? Discuss their stability.

   Answer:

   (a) Suppose $dx/dt = kx - x^3 = P(x)$. $P(x) = 0 \Rightarrow x(k - x^2) = 0$.
   If $k \leq 0$, then $k - x^2 < 0$ when $k \neq 0$. So $P(x) = 0$ iff $x = 0$, i.e., 0 is the only critical point. $P'(0) = k$. So if $k < 0$, then $P'(0) < 0 \Rightarrow x = 0$ is a stable critical point. If $k = 0$, then $P(x) > 0$ when $x < 0$ and $P(x) < 0$ when $x > 0$, so $x = 0$ is stable.

   (b) If $k > 0$, then $P(x) = 0 \Rightarrow x = 0, x = \sqrt{k}, x = -\sqrt{k}$, so there are three critical points.
   $P'(0) = k > 0$, so 0 is unstable.
   $P'(\sqrt{k}) = P'(-\sqrt{k}) = -2k < 0$, so $\sqrt{k}, -\sqrt{k}$ are stable.

2. Consider the differential equation $dx/dt = x + kx^3$ containing the

   (a) If $k \geq 0$, show that there is only one critical point. What is that point? Discuss its stability.

   (b) If $k < 0$, show that there are three critical points. What are they? Discuss their stability.

   Answer:

(a) Suppose $dx/dt = x + kx^3 = P(x)$. $P(x) = 0 \Rightarrow x(1 + kx^2) = 0$.
If $k \geq 0$, then $1 + kx^2 > 0$. So $P(x) = 0$ iff $x = 0$, i.e., 0 is the only critical point.
Since $P'(0) = 1 > 0$, 0 is unstable.

(b) If $k < 0$, then $P(x) = 0 \Rightarrow x = 0, x = \sqrt{-1/k}, x = -\sqrt{-1/k}$, so there are three critical points.
$P'(0) = 1 > 0$, so 0 is unstable.
$P'(\sqrt{-1/k}) = P'(-\sqrt{-1/k}) = -2 < 0$, so $\sqrt{-1/k}, -\sqrt{-1/k}$ are stable.