Dynamical Systems (550.391)
Homework 1 (Due Thursday, September 22, 2005)

General Directions: You must show all work and document any assumptions to receive full credit on a problem.

Part I: Population Models

1. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long will it take until all the fish are dead?

2. A tumor may be regarded as a population of multiplying cells. It is found empirically that the "birth rate" of the cells in a tumor decreases exponentially with time so that $\beta(t) = \beta_0 e^{-\alpha t}$ (where $\alpha, \beta_0 > 0$, constant). Hence

$$\frac{dP}{dt} = \beta_0 e^{-\alpha t} P, \qquad P(0) = P_0.$$

   Solve this differential equation. What is $\lim_{t\to\infty} P(t)$?

Part II: Critical Points and the Phase Portrait

The problems below involve equations of the form $dx/dt = f(x)$. For each problem, sketch the graph of $f(x)$ versus x, determine the fixed (critical) points, classify each point as stable, unstable, or semistable, and draw the phase line.

1. $$\frac{dx}{dt} = -(x-1)^2$$

2. $$\frac{dx}{dt} = x^2(x^2-1)$$

3. $$\frac{dx}{dt} = x(1-x^2)$$

Part III: Solving First-Order ODEs

For each of the functions presented in Part II, find a general solution. Sketch several graphs of particular solutions for each ODE in tx-plane. (Please solve the equation by hand. Feel free to use computer software to create your graphs.)

Hint: You may find it helpful to setup the following partial fractions problems:

$$\frac{1}{x^2(x^2-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$$

and

$$\frac{1}{x(1-x^2)} = \frac{a}{x} + \frac{b}{1-x} + \frac{c}{1+x}.$$

Part IV: Bifurcations

1. Consider the differential equation $dx/dt = kx - x^3$.

   (a) If $k \leq 0$, show that there is only one critical point. What is that point? Discuss its stability.

   (b) If $k > 0$, show that there are three critical points. What are they? Discuss their stability.

2. Consider the differential equation $dx/dt = x + kx^3$ containing the

   (a) If $k \geq 0$, show that there is only one critical point. What is that point? Discuss its stability.

   (b) If $k < 0$, show that there are three critical points. What are they? Discuss their stability.