Solutions

550.252 Practice Problems for Exam III

1. Mother Smith's Pies produces frozen fruit pies for sale to local restaurants. Demand for one of its best sellers, apple, averages 150 pies per day. The company's production facility is capable of producing 50 pies per hour and operates 8 hours a day, 365 days a year. The pies cost \$2.25 to produce and Mother Smith's estimates that its annual holding cost rate is 20%. Whenever a new pie type is produced the machinery must be thoroughly cleansed so the cost of a production setup is estimated to be \$90.

   (a) What is Q^*.

$$Q^* = \sqrt{\frac{2DC_o}{HC\left(1-\frac{D}{P}\right)}} = \sqrt{\frac{2(150)(365)(90)}{.2(2.25)\left(1-\frac{150(365)}{50(8)(365)}\right)}}$$

$D = 150 \times 365/yr$

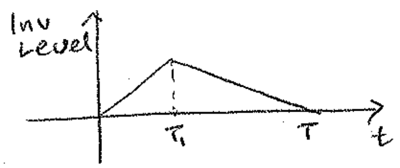
$P = 50 \times 8 \times 365/yr$

$C = \$2.25$

$H = .2$

$C_o = 90$

   (b) The time between starts of any two apple pie production runs can be divided into two distinct time periods. Explain what these two pieces represent and how you would calculate the length of each.



$[0, T_1]$: Production & Sell straight from Prod Line

$[T_1, T]$: No production, sell from inventory

$DT = Q = PT_1$ so $\boxed{T_1 = \frac{Q}{P}; \quad T - T_1 = \frac{Q}{D} - \frac{Q}{P}}$

   (c) When does the inventory reach its highest level? The highest inventory level is what fraction of Q^*?

max inventory at time T_1.

max inventory = amt produced − amt demanded during $[0, T_1]$

$$= Q - DT_1$$

$$= Q - Q\left(\frac{D}{P}\right) = Q\left(1 - \frac{D}{P}\right)$$

So max inventory is $\left(1-\frac{D}{P}\right) \times 100\%$ of Q^*

2. *Business Daily* sells its newspapers for \$.75 in coin-operated kiosks, each of which is designed to hold up to 60 papers. The production and delivery costs of each newspaper equal \$.60, but *Business Daily* receives approximately \$.23 in advertising revenue for each paper sold. Any newspapers that remain at the end of the day are sold to a recycling center. *Business Daily* estimates that the revenue it receives from the recycling center will only cover the cost of transporting the papers to the center.

Management at *Business Daily* estimates that the goodwill cost of not having enough papers in the kiosk to satisfy customer demand is \$1.50 per unsatisfied customer. Demand for the Thursday paper at a particular kiosk is estimated to follow a Poisson distribution with a mean $\lambda = 36$ units.

Hint: The Poisson distribution can be approximated by a normal distribution with $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$.)

$p = 0.98$ $c = 0.6$ $s = 0$ (no revenue or cost to salvage)

$(p = .75 + .23)$ $g = 1.50$ $\mu = \lambda = 36$ $\sigma = \sqrt{\lambda} = 6$

Select Q^* so $\Pr(D \le Q^*) = \frac{p-c+g}{p-s+g} = \frac{.98-.6+1.5}{.98-0+1.5}$

$= \frac{1.88}{2.48} = 0.7581$

$\Rightarrow \Pr\left(Z \le \frac{Q^*-\mu}{\sigma}\right) = 0.7581$ (so $Q^* > \mu$)

$0.5 + \Pr\left(0 \le Z \le \frac{Q^*-\mu}{\sigma}\right) = 0.7581$

$\Pr\left(0 \le Z \le \frac{Q^*-\mu}{\sigma}\right) = 0.2581$

$\Rightarrow \frac{Q^*-36}{6} \approx 0.7$ so $Q^* \approx 40.2$

produce either 40 or 41 newspapers daily

3. Solve the following nonlinear programming problem by first using the constraint to solve for one of the variables and then substituting to obtain an unconstrained optimization problem.

$$\min f(x_1, x_2, x_3) = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$
$$\text{subject to} \quad x_1 + x_2 + 2x_3 = 3$$

Then explain why this approach CANNOT be used to solve

$$\min f(x_1, x_2, x_3) = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$
$$\text{subject to} \quad x_1 + x_2 + 2x_3 \le 3$$

$x_1 = 3 - x_2 - 2x_3$ so

$$\hat{f}(x) = 9 - 8(3 - x_2 - 2x_3) - 6x_2 - 4x_3 + 2(3 - x_2 - 2x_3)^2 + 2x_2^2 + x_3^2 + 2(3 - x_2 - 2x_3)x_2 + 2(3 - x_2 - 2x_3)x_3$$

$$\frac{\partial \hat{f}}{\partial x_2} = 8 - 6 - 4(3 - x_2 - 2x_3) + 4x_2 + 2[(3 - x_2 - 2x_3) - x_2] - 2x_3$$
$$= -4 + 4x_2 + 2x_3$$

$$\frac{\partial \hat{f}}{\partial x_3} = 16 - 4 - 8(3 - x_2 - 2x_3) + 2x_3 - 4x_2 + 2[3 - x_2 - 2x_3 - 2x_3]$$
$$= -6 + 2x_2 + 10x_3$$

$$\begin{aligned} 4x_2 + 2x_3 &= 4 \\ 2x_2 + 10x_3 &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4x_2 + 2x_3 &= 4 \\ -4x_2 - 20x_3 &= -12 \\ \hline -18x_3 &= -8 \end{aligned} \qquad \boxed{\begin{aligned} x_3 &= 4/9 \\ x_2 &= 7/9 \\ x_1 &= 12/9 \end{aligned}}$$

$$\text{Hessian} = \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix} \quad 4 > 0, \; 10 > 0$$

and $\det(\text{Hessian}) = 40 - 4 = 36$ so This point is a global min

Substitutions only work when we have equations.

4. The temperatures (in centigrade) measured at various points inside a heated wall are as follows (with distance measured from the heated surface as a percentage of wall thickness):

Distance, d	0	25	50	75	100
Temperature, t	380	200	100	20	0

It is decided to approximate this table by a linear equation of the form $t = a + bd$ where a and b are constants, yet to be determined. Find the values of the constants a and b so as to minimize the sum of the squared difference between the approximated temperatures and the the true values.

$$\min f(a,b) = (a+0b-380)^2 + (a+25b-200)^2 + (a+50b-100)^2 + (a+75b-20)^2 + (a+100b-0)^2$$

$$\frac{\partial f}{\partial a} = 2\left[a-380+a+25b-200+a+50b-100+a+75b-20+a+100b\right]$$

$$\frac{\partial f}{\partial b} = 2\left[25(a+25b-200)+50(a+50b-100)+75(a+75b-20) + 100(a+100b-0)\right]$$

simplifying and setting $\frac{\partial f}{\partial a} = 0$ $\frac{\partial f}{\partial b} = 0$ yields

$$50a + 250b = 700$$
$$250a + 18750b = 1150$$

$$\begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} 5 & 250 \\ 250 & 18750 \end{bmatrix}^{-1} \begin{bmatrix} 700 \\ 11500 \end{bmatrix} = \begin{bmatrix} 328 \\ -3.76 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial a^2} = 10 > 0 \qquad \frac{\partial^2 f}{\partial b^2} = 2(18750) > 0 \qquad \frac{\partial^2 f}{\partial a \partial b} = \frac{\partial^2 f}{\partial b \partial a} = 2(250)$$

$$\det(\text{Hessian}) = 10(2)(18750) - [2(250)]^2 = 125000 > 0$$

so f is convex and (a^*, b^*) is a global min time measured in hours throughout

5. Ernest D. Johnson is a discount stock brokerage firm located in Sun Valley, Idaho. Johnson is the only employee, and he works out of his house where he receives all customer orders by telephone. Johnson receives an average of 10 client calls per hour, and he spends an average of 4 minutes speaking to each client. If a client calls while Johnson is talking to another client, the caller is placed on hold. The call-waiting system Johnson has leased from the telephone company allows him to keep an unlimited number of callers on hold, and he believes no customer placed on hold will hang up before talking to him. If customer calls arrive according to a Poisson distribution and the time required to provide service to a customer follows an exponential distribution, determine:

   (a) The probability that Johnson will receive exactly 4 calls in a half hour.
   (b) The probability Johnson will receive 3 or fewer calls in a half hour.
   (c) The probability that a telephone conversation will last less than 3 minutes (from the time Johnson begins speaking with the client).
   (d) The average number of callers waiting on hold.
   (e) The average time a caller will spend on the phone when calling Johnson.
   (f) The percentage of time Johnson is on the phone.
   (g) The probability that a customer will spend more than six minutes on the phone when calling Johnson.

(a) $Pr(X=4) = \frac{5^4 e^{-5}}{4!} = 0.175$ $\quad (\theta = \lambda t \text{ w/ } \lambda = 10 \ t = 1/2)$

(b) $Pr(X \le 3) = \sum_{k=0}^{3} Pr(X=k) = \sum_{k=0}^{3} \frac{5^k e^{-5}}{k!}$

$= 0.0067 + 0.0337 + 0.0842 + 0.1404 = .265$

(c) $Pr(T \le 3/60) = 1 - e^{-(60/4)(3/60)} = 1 - e^{-3/4} = 0.5276$ $\quad (\mu = 60/4 \text{ per hr})$

(d) This is L_q for an M/M/1 queuing model

$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(10)^2}{15(15-10)} = \frac{100}{75} = \frac{4}{3}$

(e) This is W for an M/M/1 queuing model $W = \frac{1}{\mu-\lambda} = .2$ hrs

(f) This is ρ for an M/M/1 queuing model $\rho = \lambda/\mu = 66.67\%$ of the time

(g) $Pr(W \ge 6/60) = e^{-(\mu-\lambda)6/60} = e^{-30/60} = 0.6065$
(Time in system for M/M/1 queue)

6. The promoter of a rock concert in Indianapolis must perform the tasks shown in the table below before the concert can be held (all durations are in days).

Activity	Description	Predecessors	Min	Most Likely	Max
A.	Find Site	–	2	3	4
B.	Find Engineers	A	1	2	3
C.	Hire opening act	A	2	6	10
D.	Set radio and TV ads	C	1	2	3
E.	Set up ticket agents	A	1	3	5
F.	Prepare electronics	B	2	3	4
G.	Print Advertising	C	3	5	7
H.	Set up transportation	C	0.5	1	1.5
I.	Rehearsals	F,H	1	1.5	2
J.	Last-minute details	I	1	2	3

(a) Draw a PERT/CPM (AON) network for this project.

(b) Find the minimum expected project duration.

(c) Find a critical path for the project.

(d) Suppose that the promoter would like to finish the project within 15 days. What is the probability of this occurring?

(e) The promoter would like to consider some options for improving the probability of completing the project within 15 days. They are

Option I	Use opening act from last performance
Option II	Contract out ad printing

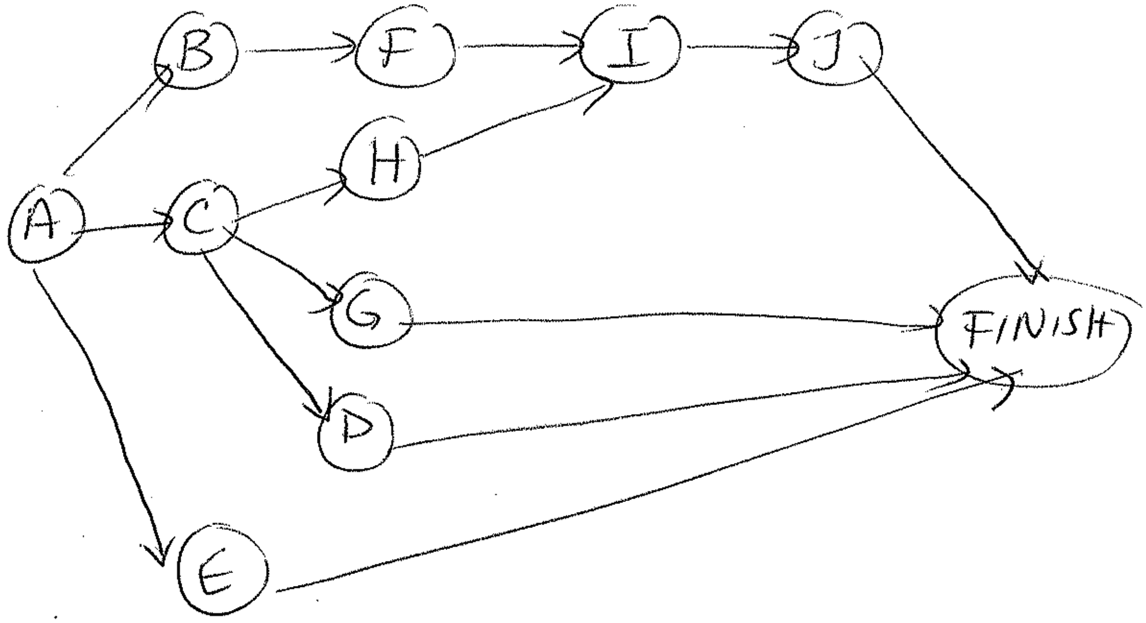
If the promoter uses Option I, this reduces the min, mode, and max times for activity C to 2, 3, and 5 respectively. The cost for this option will be \$5000 over the original cost for hiring an opening act.

If the promoter uses Option II, this reduces the min, mode, and max times for activity G to 3, 4, and 4.5 respectively. The cost for this option will be \$2000 over the original cost for ad printing.

The promoter could also choose to implement both Options I and II.

What should the promoter do?

(a)



(b) & (c)

Act ID	mean	σ	ES	EF	LS	LF	Slack
A	3	1/3	0	3	0	3	0 *
B	2	1/3	3	5	5.5	7.5	2.5
C	6	4/3	3	9	3	9	0 *
D	2	1/3	9	11	12	14	3
E	3	2/3	3	6	11	14	8
F	3	1/3	5	8	7.5	10.5	2.5
G	5	2/3	9	14	9	14	0 *
H	1	1/6	9	10	9.5	10.5	0.5
I	1.5	1/6	10	11.5	10.5	12	0.5
J	2	1/3	11.5	13.5	12	14	0.5

Critical path: $A \to C \to G$

mean project duration $\mu_{proj} = \mu_A + \mu_C + \mu_G = 14$ days.

$\sigma^2_{proj} = \sigma^2_A + \sigma^2_C + \sigma^2_G = (1/3)^2 + (4/3)^2 + (2/3)^2 = \frac{21}{9}$

(d) project duration $T \sim N(14, \frac{21}{9})$

$Pr(T \le 15) = 0.5 + Pr(0 \le Z \le \frac{15-14}{\sqrt{21/9}})$

$= 0.5 + Pr(0 \le Z \le \frac{3}{\sqrt{21}})$

$\approx 0.5 + Pr(0 \le Z \le .6547)$

$\approx 0.5 + .2422$ (using table)

$= 0.7422$

Comment: a better estimate of $Pr(0 \le Z \le .6547)$ is $\frac{1}{2}\left[Pr(0 \le Z \le .65) + Pr(0 \le Z \le .66)\right] = .7438$

$\boxed{\text{True } Pr(T \le 15) = 0.7437.}$ This value will be used in the remaining analyses

(e) <u>Option I</u>: This changes the critical path to $A \to B \to F \to I \to J$

$\mu_{opt1} = 11.5$ days $\sigma^2_{opt1} = .8056$

$Pr(T_{opt1} \le 15) = .99995$

<u>Option II</u>: This changes the critical path to $A \to C \to H \to I \to J$ $\mu_{opt2} = 13.5$ days $\sigma^2_{opt2} = 1.9514$

$Pr(T_{opt2} \le 15) = 0.8585$

Option III : This changes the critical path to

$A \to B \to F \to I \to J \quad \mu_{opt3} = 11.5 \quad \sigma^2_{opt3} = .4236$

$Pr(T \le 15) = 1$

Even w/o evaluating costs, Opt 1 would be preferred over Opt 3.

The promoter must determine what costs would be incurred if the project is not completed w/in 15 days. Based on this info, he can decide whether to stick w/ the status quo or implement either Opt 1 or Opt 2.