Stochastic Math Models (550.252)
Homework 1 (Due Friday, September 09, 2010)

General Directions: You must show all work and document any assumptions to receive full credit. When formulating models, make sure to define your variables and label your objective function and constraints (if any). All problems are to be done by hand unless otherwise stated.

Reading: In general, the course schedule (available on the website) will list the reading associated with each assignment. However, since this assignment uses a number of handouts, the readings are included here in detail. All handouts are available on the course website.

- *Introduction to Probability* Handout Sections 2.1-2.3 (up to page 70).
- *Basic Linear Algebra, Part 1* Handout Sections 2.2-2.2

Problems:

1. Three marbles are drawn successively and at random from a box containing 10 red, 30 white, 20 blue, and 15 orange marbles. Determine the probability that they are drawn in the order red, white, blue if

   (a) each marble is replaced before the next is drawn.

   (b) each marble is NOT replaced before the next is drawn.

   Make sure you explain your answers clearly.

2. Let n be a nonnegative integer less than 365. Find the probability than n people selected at random will have n different birthdays.

3. We would like to find the n such that the birthday probability you developed in Problem 2 is less than 0.50. To do this

   (a) Denote the probability you developed in Problem 2 as p. Find an expression for $\ln p$ that is itself a sum of natural logs.

   (b) From calculus, we know that

   $$-\frac{d}{dx}\ln(1-x) = \frac{1}{1-x}$$

   and that if $|x| < 1$ then

   $$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}.$$

   Using these two pieces of information, create an infinite sum expression for $\ln(1-x)$.

(c) Use the result of part (b) to explain why

$$\ln p = -\left[\frac{1+2+\cdots+(n-1)}{365}\right] - \frac{1}{2}\left[\frac{1^2+2^2+\cdots+(n-1)^2}{(365)^2}\right] - \frac{1}{3}\left[\frac{1^3+2^3+\cdots+(n-1)^3}{(365)^3}\right] - \cdots$$

(d) When $n < 30$, the second and higher order terms in the above expression are negligible when compared to the first term. Using this fact and the fact that

$$1+2+\cdots+(n-1) = \frac{n(n-1)}{2}$$

show that if n is 23 or larger then there are better than even odds that at least two people will have the same birthday.

4. Consider the system of equations presented below.

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}$$

(a) Write the system as a matrix equation.

(b) Solve the system (BY HAND!) using any approach (e.g., substitution, addition/subtraction, Gaussian elimination, etc.).

5. Compute (BY HAND!) $\mathbf{A} + \mathbf{B}$ and $\mathbf{AC}$ where

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{bmatrix}.$$

6. Consider a metropolitan area with a constant total population of 1 million individuals. This area consists of a city and its suburbs. Let $C(k)$ denote the city population and $S(k)$ denote the suburb population after k years. Suppose that each year, 15% of the people in the city move to the suburbs and 10% of the people in the suburbs move to the city. Create a system of linear equations to relate the area's population in year $k+1$ to the population in year k.