Stochastic Math Models (550.252) Exam I

Friday, October 07, 2011

General Directions: This exam is closed book, NO calculator. You may use one 8.5x11 sheet of notes. To receive credit for a problem you must SHOW ALL WORK and write neatly and clearly. Please use the space provided. If you require extra space, please feel free to write on the back or attach extra sheets.

PRINT Name Solutions

Problem	*Points Earned*	*Points Possible*
1		20
2		15
3		25
4		25
5		15
Total		100

I attest that I have completed this exam without unauthorized assistance from any person, materials, or device.

Please SIGN and DATE in INK.

1. Let the pmf of X be

$$p(x) = \begin{cases} \frac{x}{10} & x = 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $Var(X)$.

(b) Find the cdf for X.

(a) $$E(X) = \sum_{x=1}^{4} x\,p(x) = 1\left(\frac{1}{10}\right) + 2\left(\frac{2}{10}\right) + 3\left(\frac{3}{10}\right) + \frac{4}{10}(4) = \frac{1+4+9+16}{10} = \frac{30}{10}$$

$$E(X^2) = \sum_{x=1}^{4} x^2 p(x) = 1\left(\frac{1}{10}\right) + 4\left(\frac{2}{10}\right) + 9\left(\frac{3}{10}\right) + 16\left(\frac{4}{10}\right) = \frac{1+8+27+64}{10} = \frac{100}{10} = 10$$

$$Var(X) = E(X^2) - (E(X))^2 = 10 - 9 = 1$$

(b)

x	$p(x)$	$F(x) =$
1	$1/10$	$1/10$
2	$2/10$	$3/10$
3	$3/10$	$6/10$
4	$4/10$	1

cdf

2. Find the mean times to absorption for the Markov process with the following transition matrix.

$$\begin{array}{c} \text{State} = \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & & \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0.10 & 0.20 & 0.50 & 0.20 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{array}\right] \end{array} = \left[\begin{array}{c|c} I_{abs} & O \\ \hline R & Q \end{array}\right]$$

$$N = (I_Q - Q)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 2/10 \\ 1/4 & 1/4 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1/2 & -1/5 \\ -1/4 & 3/4 \end{bmatrix}^{-1} = \frac{1}{\frac{1}{2}\left(\frac{3}{4}\right) - \frac{1}{4}\left(\frac{1}{5}\right)} \begin{bmatrix} 3/4 & 1/5 \\ 1/4 & 1/2 \end{bmatrix}$$

$$= \frac{1}{\frac{3}{8} - \frac{1}{20}} \begin{bmatrix} 3/4 & 1/5 \\ 1/4 & 1/2 \end{bmatrix}$$

For state 1: $\frac{1}{\frac{3}{8} - \frac{1}{20}} \left(\frac{3}{4} + \frac{1}{5} \right)$

For state 2: $\frac{1}{3/8 - \frac{1}{20}} \left(3/4 \right)$

3. Let the pdf of X be

$$f(x) = \begin{cases} \frac{1}{6}(x^3 + x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $E(X)$?

(b) What is $F(x)$?

(c) What is $Pr(X > 0.5)$?

(d) What is $Pr(X > 1 | X > 0.5)$?

(e) Is X *memoryless*? Explain.

(a) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{6}\int_0^2 x(x^3+x)\,dx = \frac{1}{6}\left[\frac{1}{5}x^5 + \frac{1}{3}x^3\right]_0^2$

$= \frac{1}{6}\left(\frac{32}{5} + \frac{8}{3}\right)$

(b) $F(x) = \int_{-\infty}^{x} f(u)\,du = \frac{1}{6}\int_0^x (u^3+u)\,du = \frac{1}{6}\left[\frac{1}{4}u^4 + \frac{1}{2}u^2\right]_0^x$

$= \frac{1}{6}\left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) = \frac{1}{24}x^4 + \frac{1}{12}x^2$

$$= \begin{cases} 0 & x < 0 \\ \frac{1}{24}x^4 + \frac{1}{12}x^2 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

(c) $Pr(X > 0.5) = 1 - F(0.5) = 1 - \left[\frac{1}{24}\left(\frac{1}{2}\right)^4 + \frac{1}{12}\left(\frac{1}{2}\right)^2\right] = 1 - \frac{9}{24(16)} = \frac{125}{128}$

(d) $Pr(X>1 \mid X>\frac{1}{2}) = \frac{Pr(X>1 \cap X>\frac{1}{2})}{Pr(X>\frac{1}{2})} = \frac{Pr(X>1)}{Pr(X>\frac{1}{2})} = \frac{1-F(1)}{1-F(\frac{1}{2})}$

$= \frac{1-\left[\frac{1}{24}+\frac{1}{12}\right]}{1-\frac{9}{24(16)}} = \frac{7/8}{125/128} = \frac{112}{125}$

(e) No. If it were, we'd have $Pr(X>1 \mid X>\frac{1}{2}) = Pr(X>\frac{1}{2})$ (as for the exponential distribution)

4. Find the steady state probabilities for the Markov process with the following transition matrix.

$$\begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.50 & 0 & 0.50 \\ 0.25 & 0.50 & 0.25 \end{bmatrix}$$

$\pi^* = \pi^* P \qquad \sum \pi_i^* = 1$

$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3$

$\pi_2 = \frac{1}{4}\pi_1 + 0\pi_2 + \frac{1}{2}\pi_3$

$\Rightarrow$

$-\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = 0$

$\frac{1}{4}\pi_1 - \pi_2 + \frac{1}{2}\pi_3 = 0$

$\pi_1 + \pi_2 + \pi_3 = 1$

substitute for π_2

$-\frac{1}{2}\pi_1 + \frac{1}{2}\left(\frac{1}{4}\pi_1 + \frac{1}{2}\pi_3\right) + \frac{1}{4}\pi_3 = 0$

$-\frac{3}{8}\pi_1 + \frac{1}{2}\pi_3 = 0$

$\pi_3 = \frac{3}{4}\pi_1$

$\boxed{\pi_3^* = \frac{3}{\cancel{4}} \cdot \frac{\cancel{8}^{2}}{19} = \frac{6}{19}}$

$\pi_1 + \left(\frac{1}{4}\pi_1 + \frac{1}{2}\pi_3\right) + \pi_3 = 1$

$\frac{5}{4}\pi_1 + \frac{3}{2}\pi_3 = 1$

$\frac{5}{4}\pi_1 + \frac{3}{2}\left(\frac{3}{4}\pi_1\right) = 1$

$\frac{19}{8}\pi_1 = 1$

$\boxed{\pi_1^* = \frac{8}{19}}$

$\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \boxed{\pi_2^* = \frac{5}{19}}$

5. Short Answer Questions

(a) If customers arrive at a shop according to a Poisson process with a mean rate λ what is the probability distribution for the waiting time until the first arrival?

exponential with mean $1/\lambda$

(b) Customers arrive at a shop according to a Poisson process at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer? $\lambda = 20/hr = 1/3\ min$

$Pr(X > 5) = 1 - Pr(X < 5) = 1 - (1 - e^{-u(5)}) = e^{-15}$

where $u = 3$

(c) What is the difference between a stochastic mathematical model and a deterministic mathematical model?

stochastic models assume some or all data are uncertain (and follow some known or unknown probabilistic distribution)

deterministic models assume all data a known with certainty.