36. A certain radioactive substance is decaying according to the equation

$$\frac{dA}{dt} = -0.25A,$$

where A is the amount of substance in milligrams remaining after t days. Suppose that the initial amount of the substance present is 400 mg. Use a numerical solver to estimate the amount of substance remaining after 4 days.

37. The concentration of pollutant in a lake is given by the equation

$$\frac{dc}{dt} = -0.055c,$$

where c is the concentration of the pollutant at t days. Suppose that the initial concentration of pollutant is 0.10. A concentration level of $c = 0.02$ is deemed safe for the fish population in the lake. If the concentration varies according to the model, how long will it be before the concentration reaches a level that is safe for the fish population?

38. An aluminum rod is heated to a temperature of 300°C. Suppose that the rate at which the rod cools is proportional to the difference between the temperature of the rod and the temperature of the surrounding air (20°C). Assume a proportionality constant $k = 0.085$ and time is measured in minutes. How long will it take the rod to cool to 100°C?

39. You're told that the "carrying capacity" for an environment populated by "critters" is 100. Further, you're also told that the rate at which the critter population is changing is proportional to the product of the number of critters and the number of critters less than the carrying capacity. Assuming a constant of proportionality $k = 0.00125$ and an initial critter population of 20, use a numerical solver to determine the size of the critter population after 30 days.

2.2 Solutions to Separable Equations

An unstable nucleus is radioactive. At any instant, it can emit a particle, transforming itself into a different nucleus in the process. For example, ^{238}U is an alpha emitter that decays spontaneously according to the scheme $^{238}U \to {}^{234}Th + {}^{4}He$, where ^{4}He is the alpha particle. In a sample of ^{238}U, a certain percentage of the nuclei will decay during a given observation period. If at time t the sample contains $N(t)$ radioactive nuclei, then we expect that the number of nuclei that decay in the time interval Δt will be approximately proportional to both N and Δt. In symbols,

$$\Delta N = N(t + \Delta t) - N(t) \approx -\lambda N(t)\Delta t, \tag{2.1}$$

where $\lambda > 0$ is a constant of proportionality. The minus sign is indicative of the fact that there are fewer radioactive nuclei at time $t + \Delta t$ than there are at time t.

Dividing both sides of equation (2.1) by Δt, then taking the limit as $\Delta t \to 0$,

$$N'(t) = \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{\Delta t} = -\lambda N(t).$$

This equation is one that arises often in applications. Because of the form of its solutions, the equation

$$N' = -\lambda N \tag{2.2}$$

is called the ***exponential equation***.

Equation (2.2) is an example of what is called a ***separable equation*** because it can be rewritten with its variables separated and then easily solved. To do this, we first write the equation using dN/dt instead of N',

$$\frac{dN}{dt} = -\lambda N. \tag{2.3}$$

Next, we separate the variables by putting every expression involving the unknown function N on the left and everything involving the independent variable t on the right. This includes dN and dt. The result is

$$\frac{1}{N}\,dN = -\lambda\,dt. \tag{2.4}$$

It is important to note that this step is valid only if $N \neq 0$, since we cannot divide by zero. Then we integrate both sides of equation (2.4), getting[2]

$$\int \frac{1}{N}\,dN = -\lambda \int dt, \quad \text{or}$$
$$\ln|N| = -\lambda t + C. \tag{2.5}$$

It remains to solve for N. Taking the exponential of both sides of equation (2.5), we get

$$|N(t)| = e^{-\lambda t + C} = e^C e^{-\lambda t}. \tag{2.6}$$

Since e^C and $e^{-\lambda t}$ are both positive, there are two cases

$$N(t) = \begin{cases} e^C e^{-\lambda t}, & \text{if } N > 0; \\ -e^C e^{-\lambda t}, & \text{if } N < 0. \end{cases}$$

We can simplify the solution by introducing

$$A = \begin{cases} e^C, & \text{if } N > 0; \\ -e^C, & \text{if } N < 0. \end{cases}$$

Therefore, the solution is also described by the simpler formula

$$N(t) = Ae^{-\lambda t}, \tag{2.7}$$

where A is a constant different from zero, but otherwise arbitrary.

In arriving at equation (2.4), we divided both sides of equation (2.3) by N, and this procedure is not valid when $N = 0$. We will discuss this a bit later. For now, let's notice that if we set $A = 0$ in equation (2.7), we get the constant function $N(t) = 0$, and we can verify by substitution that this is a solution of the original equation, $N' = -\lambda N$. Consequently, equation (2.7) with A completely arbitrary, gives us the solution in all cases.

Example 2.8 ^{32}P, an isotope of phosphorus, is used in leukemia therapy. After 10 hours, 615 mg of an initial 1000 mg sample remain. The ***half-life*** of a radioactive substance is the amount of time required for 50% of the substance to decay. Determine the half-life of ^{32}P.

The differential equation $N' = -\lambda N$ was used to model the number of remaining nuclei. However, the number of nuclei is proportional to the mass, so we will let N represent the mass of the remaining nuclei in this example. As seen earlier, this differential equation has solution

$$N = Ae^{-\lambda t}, \tag{2.9}$$

where A is an arbitrary constant. At time $t = 0$ we have $N = 1000$ mg of the isotope. Substituting these quantities in equation (2.9),

$$1000 = Ae^{-\lambda(0)} = A. \tag{2.10}$$

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[2] Our understanding of integration first has us use two constants of integration,

$$\ln|N| + C_1 = -\lambda t + C_2.$$

We get (2.5) by setting $C = C_2 - C_1$. This combining of the two constants into one works in the solution of any separable equation.

Consequently, equation (2.9) becomes

$$N = 1000e^{-\lambda t}. \tag{2.11}$$

After $t = 10\,\text{hr}$, only $N = 615\,\text{mg}$ of the substance remains. Substituting these values into equation (2.11), we get

$$615 = 1000e^{-\lambda(10)}. \tag{2.12}$$

Using a little algebra and a calculator to compute a logarithm shows that $\lambda = 0.04861$, correct to six decimal places, and equation (2.11) becomes

$$N = 1000e^{-0.04861t}. \tag{2.13}$$

To find the half-life, we substitute $N = 500\,\text{mg}$ in equation (2.13).

$$500 = 1000e^{-0.04861t}$$

Solving for t, we find that the half-life of the isotope is approximately 14.3 hours. ●

A large number of equations are separable and can be solved exactly like we solved the exponential equation. Let's look at another example.

Example 2.14 Solve the differential equation

$$y' = ty^2. \tag{2.15}$$

Again, we rewrite the equation using dy/dt instead of y', so

$$\frac{dy}{dt} = ty^2. \tag{2.16}$$

Next we separate the variables by putting every expression involving the unknown function y on the left and everything involving the independent variable t on the right, including dy and dt. The result is

$$\frac{1}{y^2}\,dy = t\,dt. \tag{2.17}$$

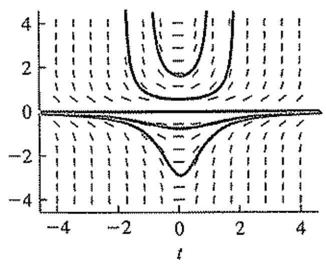

Figure 1. Several solutions to $y' = ty^2$.

Notice that this step is valid only if $y \neq 0$, since we cannot divide by zero. Next we integrate both sides of equation (2.17), getting

$$\int \frac{1}{y^2}\,dy = \int t\,dt, \quad \text{or} \quad -\frac{1}{y} = \frac{1}{2}t^2 + C. \tag{2.18}$$

Finally, we solve equation (2.18) for y. The equation for the solution is

$$y(t) = \frac{-1}{\frac{1}{2}t^2 + C} = \frac{-2}{t^2 + 2C}. \tag{2.19}$$

Several solutions are shown in Figure 1. Included among the functions plotted in Figure 1 is the constant function $y(t) = 0$. It is easily verified by substitution that this is a solution of (2.15), although no finite value of C in equation (2.19) will yield this solution. We will have more to say about this on page 30. ●

Treating dy and dt as mathematical entities, as we did in separating the variables in equation (2.17), may be troublesome to you. If so, it is probably because you have learned your calculus very well. We will explain this step at the end of this section under the heading "Why separation of variables works."

The general method

Clearly the key step in this method is the separation of variables. This is the step going from equation (2.3) to equation (2.4) or from equation (2.16) to equation (2.17). The method of solution illustrated here will work whenever we can perform this step, and this can be done for any equation of the two equivalent forms

$$\frac{dy}{dt} = \frac{g(t)}{h(y)} \tag{2.20}$$

and

$$\frac{dy}{dt} = g(t)f(y). \tag{2.21}$$

Equations of either form are called ***separable*** differential equations. For both we can separate the variables.

The method we used to solve equation (2.16) will work for any separable equation.

We can solve any separable equation of the form (2.21) using the following three steps.

1. Separate the variables: $\dfrac{dy}{f(y)} = g(t)\,dt$.
2. Integrate both sides: $\displaystyle\int \frac{dy}{f(y)} = \int g(t)\,dt$.
3. Solve for the solution $y(t)$, if possible.

Avoiding division by zero

When separating the variables we do have to worry about dividing by zero, but otherwise things work well. What about those points where $f(y) = 0$ in equation (2.21)? It turns out to be quite easy to find the solutions in such a case, since if $f(y_0) = 0$, then by substitution we see that the constant function $y(t) = y_0$ is a solution to (2.21).

In particular, the function $y(t) = 0$ is a solution to the equation $y' = ty^2$. We found in (2.19) that, under the assumption that $y \neq 0$, the general solution to the equation $y' = ty^2$ is

$$y(t) = \frac{-2}{t^2 + 2C}.$$

If we naively substitute the initial condition $y(0) = 0$ into this general solution, we get $0 = -1/C$. No finite value of the constant C solves this equation. This should not be a surprise, since (2.19) was derived on the assumption that $y \neq 0$. Nevertheless, we will want to call (2.19) a general solution to equation (2.16). We define a ***general solution*** to a differential equation to be a family of solutions depending on sufficiently many parameters to give all but finitely many solutions.

Thus the general solution to a differential equation does not always yield the solution to every initial value problem, and for separable equations this is related to the problem of dividing by 0. In the case of $y' = ty^2$, we can find the exceptional solution by setting $C = \infty$. This often the case, but we will not explore this further.

Using definite integration

Sometimes it is useful to use definite integrals when solving initial value problems for separable equations.

Example 2.22 A can of beer at 40°F is placed into a room where the temperature is 70°F. After 10 minutes the temperature of the beer is 50°F. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 minutes after the beer was placed into the room?

According to ***Newton's law of cooling***, the rate of change of an object's temperature (T) is proportional to the difference between its temperature and the ambient temperature (A). Thus we have

$$\frac{dT}{dt} = -k(T - A). \tag{2.23}$$

We introduce the minus sign so that the proportionality constant k is positive. Notice that if $T < A$, the temperature of the object will be increasing. The equation is separable, so we separate variables to get

$$\frac{dT}{T - A} = -k\,dt.$$

The next step is to integrate both sides, but this time let's use definite integrals to bring in the initial condition $T(0) = T_0$. Since $t = 0$ corresponds to $T = T_0$, we have

$$\int_{T_0}^{T} \frac{ds}{s - A} = -k \int_0^t du.$$

Notice that we changed the variables of integration because we want the upper limits of our integrals to be T and t. Performing the integration, we get

$$\ln \frac{|T - A|}{|T_0 - A|} = \ln|T - A| - \ln|T_0 - A| = -kt.$$

We can solve for T by exponentiating, and since $T - A$ and $T_0 - A$ both have the same sign, our answer is

$$T(t) = A + (T_0 - A)e^{-kt}. \tag{2.24}$$

We first use the fact that $T(10) = 50$ in addition to the initial condition $T(0) = T_0 = 40$ and the ambient temperature $A = 70$ to evaluate k. Equation (2.24) becomes $50 = 70 - 30e^{-10k}$. Therefore, $k = \ln(3/2)/10 = 0.0405$. Thus from equation (2.24) we see that the temperature is

$$T(t) = 70 - 30e^{-0.0405t}.$$

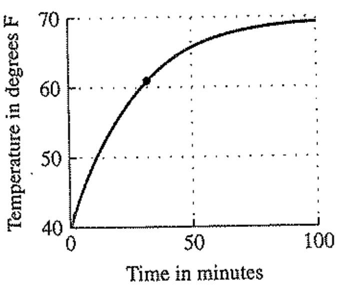

Figure 2. The temperature of the can of beer in Example 2.22.

After 30 minutes the temperature is 61.1°F. The solution is plotted in Figure 2. ◉

Implicitly defined solutions

After the integration step, we need to solve for the solution. However, this is not always easy. In fact, it is not always possible. We will look at a series of examples.

Example 2.25 Find the solutions of the equation $y' = e^x/(1+y)$, having initial conditions $y(0) = 1$ and $y(0) = -4$.

Separate the variables and integrate.

$$(1+y)\,dy = e^x\,dx$$

$$y + \frac{1}{2}y^2 = e^x + C \tag{2.26}$$

Rearrange equation (2.26) as

$$y^2 + 2y - 2(e^x + C) = 0. \tag{2.27}$$

This is an implicit equation for $y(x)$ that we can solve using the quadratic formula.

$$\begin{aligned} y(x) &= \frac{1}{2}\left[-2 \pm \sqrt{4 + 8(e^x + C)}\right] \\ &= -1 \pm \sqrt{1 + 2(e^x + C)} \end{aligned}$$

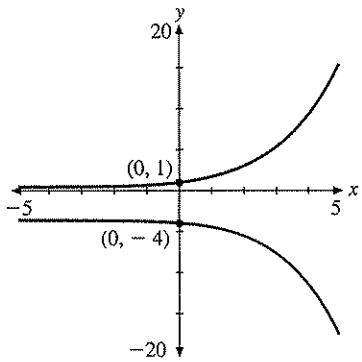

Figure 3. $y = -1 + \sqrt{2 + 2e^x}$ passes through (0, 1), while $y = -1 - \sqrt{7 + 2e^x}$ passes through (0, −4).

We get two solutions from the quadratic formula, and the initial condition will dictate which solution we choose. If $y(0) = 1$, then we must use the positive square root and we find that $C = 1/2$. The solution is

$$y(x) = -1 + \sqrt{2 + 2e^x}. \tag{2.28}$$

On the other hand, if $y(0) = -4$, then we must use the negative square root and we find that $C = 3$. The solution in this case is

$$y(x) = -1 - \sqrt{7 + 2e^x}. \tag{2.29}$$

Both solutions are shown in Figure 3.

What about the interval of existence? A quick glance reveals that each solution is defined on the interval $(-\infty, \infty)$. Some calculation will reveal that $y'(x)$ is also defined on $(-\infty, \infty)$. However, for each solution to satisfy the equation $y' = e^x/(1+y)$, y must not equal -1. Fortunately, neither solution (2.28) or (2.29) can ever equal -1. Therefore, the interval of existence is $(-\infty, \infty)$. ●

Let's be sure we know what the terminology means. An ***explicit*** solution is one for which we have a formula as a function of the independent variable. For example, (2.28) is an explicit solution to the equation in Example 2.25. In contrast, (2.27) is an implicit equation for the solution. In this example, the implicit equation can be solved easily to find an explicit equation, but this is not always the case.

Unfortunately, implicit solutions occur frequently. Consider again the general problem in the form $dy/dt = g(t)/h(y)$. Separating variables and integrating, we get

$$\int h(y)\,dy = \int g(t)\,dt. \tag{2.30}$$

If we let

$$H(y) = \int h(y)\,dy \quad \text{and} \quad G(t) = \int g(t)\,dt,$$

and then introduce a constant of integration, equation (2.30) can be rewritten as

$$H(y) = G(t) + C. \tag{2.31}$$

Unless $H(y) = y$, and therefore $h(y) = 1$, this is an implicit equation for $y(t)$. To find an explicit solution we must be able to compute the inverse function H^{-1}. If this is possible, then we have

$$y(t) = H^{-1}(G(t) + C).$$

Let's do one more example.

Example 2.32 Find the solutions to the differential equation

$$x' = \frac{2tx}{1+x},$$

having initial conditions $x(0) = 1$, $x(0) = -2$, and $x(0) = 0$.

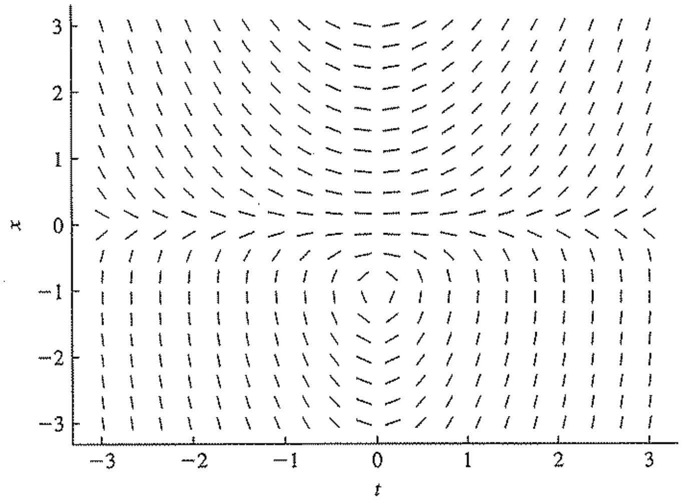

Figure 4. The direction field for $x' = 2tx(1+x)$.

The direction field for this equation is shown in Figure 4. This equation is separable since it can be written as

$$\frac{dx}{dt} = 2t\frac{x}{1+x}.$$

When we separate variables, we get

$$\left(1 + \frac{1}{x}\right) dx = 2t\,dt,$$

assuming that $x \neq 0$. Integrating, we get

$$x + \ln(|x|) = t^2 + C, \tag{2.33}$$

where C is an arbitrary constant. For the initial condition $x(0) = 1$, this becomes $1 = C$. Hence our solution is implicitly defined by $x + \ln(|x|) = t^2 + 1$. The function $\ln(|x|)$ is not defined at $x = 0$, so our solution can never be equal to 0. Since our initial condition is positive, and a solution must be continuous, our solution $x(t)$ must be positive for all t. Hence $|x| = x$ and our solution is given implicitly by

$$x + \ln(x) = t^2 + 1. \tag{2.34}$$

This is as far as we can go. We cannot solve equation (2.34) explicitly for $x(t)$, so we have to be satisfied with this as our answer. The solution x is defined implicitly by equation (2.34).

For the initial condition $x(0) = -2$, we can find the constant C in the same manner. We get $-2+\ln(|-2|) = C$, or $C = \ln 2 - 2$. Hence the solution is defined implicitly by

$$x + \ln(|x|) = t^2 + \ln 2 - 2.$$

This time our initial condition is negative, so $|x| = -x$, and our implicit equation for the solution is

$$x + \ln(-x) = t^2 + \ln 2 - 2.$$

For the initial condition $x(0) = 0$, we cannot divide by $x/(1+x)$ to separate variables. However, we know that this means that $x(t) = 0$ is a solution. We can easily verify that by direct substitution. Thus we do get an explicit formula for the solution with this initial condition. ●

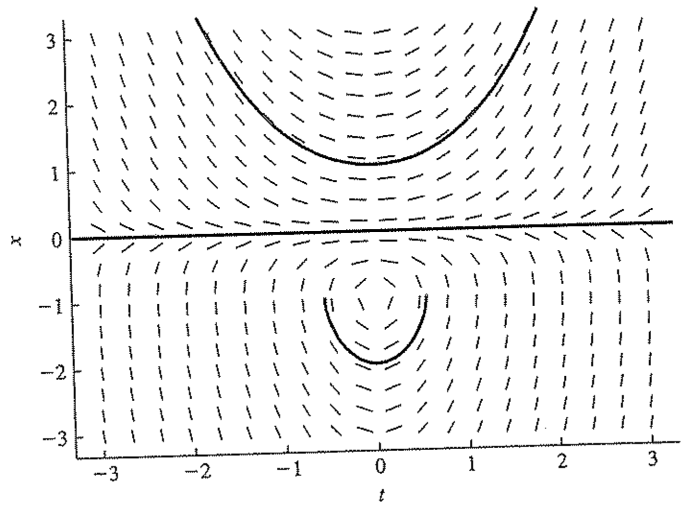

Figure 5. Solutions to $x' = 2tx(1+x)$.

The solutions sought in the previous example were computed numerically and are plotted in Figure 5. Since the solutions are defined implicitly, it is a difficult task to visualize them without the aid of numerical methods.

Why separation of variables works

If we start with a separable equation

$$y' = g(t)/h(y), \tag{2.35}$$

then separation of variables leads to the equation

$$h(y)\,dy = g(t)\,dt. \tag{2.36}$$

However, many readers will have been taught that the terms dy and dt have no meaning and so equation (2.36) has no meaning. Yet the method works, so what is going on here?

To understand this better, let's start with (2.35) and perform legitimate steps

$$y' = g(t)/h(y) \quad \text{or} \quad h(y)y' = g(t).$$

Integrating both sides with respect to t, we get

$$\int h(y(t))y'(t)\,dt = \int g(t)\,dt.$$

The integral on the left contains the expression $y'(t)\,dt$. This is inviting us to change the variable of integration to y, since when we do that, we use the equation $dy = y'(t)\,dt$. Making the change of variables leads to

$$\int h(y)\,dy = \int g(t)\,dt. \tag{2.37}$$

Notice the similarity between (2.36) and (2.37). Equation (2.36), which has no meaning by itself, acquires a precise meaning when both sides are integrated. Since this is precisely the next step that we take when solving separable equations, we can be sure that our method is valid.

We mention in closing that the objects in (2.36), $h(y)\,dy$ and $g(t)\,dt$, can be given meaning as formal objects that can be integrated. They are called ***differential forms***, and the special cases like dy and dt are called ***differentials***. The basic formula connecting differentials dy and dt when y is a function of t is

$$dy = y'(t)\,dt,$$

the change-of-variables formula in integration. These techniques will assume greater importance in Section 2.6, where we will deal with exact equations. The use of differential forms is very important in the study of the calculus of functions of several variables and especially in applications to geometry and to parts of physics.

EXERCISES

In Exercises 1–12, find the general solution of the indicated differential equation. If possible, find an explicit solution.

1. $y' = xy$

2. $xy' = 2y$

3. $y' = e^{x-y}$

4. $y' = (1 + y^2)e^x$

5. $y' = xy + y$

6. $y' = ye^x - 2e^x + y - 2$

7. $y' = x/(y+2)$

8. $y' = xy/(x-1)$

9. $x^2y' = y\ln y - y'$

10. $xy' - y = 2x^2y$

11. $y^3y' = x + 2y'$

12. $y' = (2xy+2x)/(x^2-1)$

In Exercises 13–18, find the exact solution of the initial value problem. Indicate the interval of existence.

13. $y' = y/x$, $y(1) = -2$

14. $y' = -2t(1+y^2)/y$, $y(0) = 1$

15. $y' = (\sin x)/y$, $y(\pi/2) = 1$

16. $y' = e^{x+y}$, $y(0) = 0$

17. $y' = (1 + y^2)$, $y(0) = 1$

18. $y' = x/(1+2y)$, $y(-1) = 0$

In Exercises 19–22, find exact solutions for each given initial condition. State the interval of existence in each case. Plot each exact solution on the interval of existence. Use a numerical solver to duplicate the solution curve for each initial value problem.

19. $y' = x/y$, $y(0) = 1$, $y(0) = -1$

20. $y' = -x/y$, $y(0) = 2$, $y(0) = -2$

21. $y' = 2 - y$, $y(0) = 3$, $y(0) = 1$

22. $y' = (y^2 + 1)/y$, $y(1) = 2$

23. Suppose that a radioactive substance decays according to the model $N' = N_0e^{-\lambda t}$. Show that the half-life of the radioactive substance is given by the equation

$$T_{1/2} = \frac{\ln 2}{\lambda}. \tag{2.38}$$

24. The half-life of ^{238}U is 4.47×10^7 yr.

(a) Use equation (2.38) to compute the ***decay constant*** λ for ^{238}U.

(b) Suppose that 1000 mg of ^{238}U are present initially. Use the equation $N = N_0e^{-\lambda t}$ and the decay constant determined in part (a) to determine the time for this sample to decay to 100 mg.

25. Tritium, ^{3}H, is an isotope of hydrogen that is sometimes used as a biochemical tracer. Suppose that 100 mg of ^{3}H decays to 80 mg in 4 hours. Determine the half-life of ^{3}H.

26. The isotope Technetium 99m is used in medical imaging. It has a half-life of about 6 hours, a useful feature for radioisotopes that are injected into humans. The Technetium, having such a short half-life, is created artificially on scene by harvesting from a more stable isotope, ^{99}Mb. If 10 g of ^{99m}Tc are "harvested" from the Molybdenum, how much of this sample remains after 9 hours?

27. The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 500 mg of ^{131}I, how much of the isotope will be left after 20 days?