From: Polking, Boggess & Arnold. Differential Equations w/ Boundary Value Problems, 2nd Edition

3.3 Personal Finance

There are a number of problems involving personal finance that can be modeled using differential equations. We will start by considering what happens to the balance of a savings account, or of money invested in a portfolio of stocks and bonds.

Let $P(t)$ be the balance at time t, and suppose the account pays interest at a rate of r percent per year, compounded continuously. This means that the increase in the balance between times t and $t + \Delta t$ is

$$P(t + \Delta t) - P(t) = \text{interest earned in time } \Delta t. \tag{3.1}$$

Since r is the interest rate over a year, the interest earned in time Δt is approximately

$$\text{interest earned in time } \Delta t \approx rP\Delta t. \tag{3.2}$$

Hence, our model is

$$\begin{aligned} P'(t) &= \lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{rP\Delta t}{\Delta t} \\ &= rP. \end{aligned} \tag{3.3}$$

This is once more the exponential equation, which we have already seen in a variety of applications. We know that the solutions have the form

$$P(t) = Ce^{rt}.$$

If the initial balance is $P(0) = P_0$, then we see that $C = P_0$ and

$$P(t) = P_0 e^{rt}.$$

Example 3.4 Suppose you put \$1000 in a savings account with a continuously compounded interest rate of 5% per year. What will be the balance after 40 years?

In this case, $P_0 = 1000$, and $r = 0.05$. Hence

$$P(40) = 1000e^{0.05 \times 40} = 1000e^2 \approx 7389.$$

Under the assumptions we have made, the balance in our savings account will grow exponentially. The basic assumption is that the interest rate r is a constant. Of course, in practice this is not true. Interest rates change constantly in reaction to a variety of economic and political events and are unpredictable. This limits the effectiveness of our model. To allow for this unpredictability, we should do the analysis for a variety of interest rates ranging from the lowest to the highest that we expect. This kind of "what if" analysis will allow us to bracket the real outcome.

An interest-bearing account with steady withdrawals

Next let's look at the balance $P(t)$ in a savings or an investment account from which the amount W is withdrawn every year and that pays interest at the rate of r percent per year, compounded continuously. When we add the effect of the withdrawals, equation (3.1) becomes

$$P(t + \Delta t) - P(t) = \text{interest earned in time } \Delta t - \text{withdrawal in time } \Delta t. \tag{3.5}$$

Once more, the interest earned is given by (3.2). On the other hand, if W is the amount withdrawn per unit time, the amount withdrawn in time Δt is

$$W\Delta t. \tag{3.6}$$

Thus,

$$P(t + \Delta t) - P(t) \approx rP(t)\Delta t - W\Delta t.$$

We can compute the derivative using the limit quotient definition:

$$\begin{aligned}\frac{dP}{dt} &= \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{rP(t)\Delta t - W\Delta t}{\Delta t} \\ &= rP - W.\end{aligned}$$

Hence our model is

$$\frac{dP}{dt} = rP - W. \tag{3.7}$$

On the other hand, if we deposit D dollars per year, the equation becomes

$$P' = rP + D. \tag{3.8}$$

Systematic use of dimensions and units

Another factor that can often be used to keep confusion at bay during the modeling process is the careful and systematic use of dimensions and units. For example, the difference between equations (3.2) and (3.6) can be misleading. In particular, why doesn't (3.6) read $WP\Delta t$ in complete analogy to (3.2)? The answer is that r and W have different dimensions. The interest rate r is usually referred to as a percentage. It measures the fraction of a dollar earned per unit time. Since a fraction has no dimension and we are measuring time in years, the dimensions of r are simply years^{-1}. On the other hand, the withdrawal rate W is expressed in terms of dollars per year. Hence the units of W are dollars $\times$ years^{-1}. The left-hand side of (3.5) is measured in dollars. With the dimensions of r and W as found earlier, the expressions of both (3.2) and (3.6) are also measured in dollars. On the other hand, the expression $WP\Delta t$ is measured in (dollars)^2, which does not match.

The point is that by keeping careful track of the dimensions of the quantities being used, it is possible to keep from making errors.

Systematic savings

Our next problem is to start planning for retirement. The same process models savings towards any financial goal, such as the saving for the education of a child or the purchase of a house.

Example 3.9 Suppose you are just starting to work and you decide to save \$2000 each year. Assuming that you have no savings to begin with and that your savings will earn 5% per year, compounded continuously, how much will you accumulate after 30 years?

We developed the model for this in equation (3.8). The equation in this case is

$$P' = 0.05\,P + 2,$$

where P is the principal balance in thousands of dollars. This is a linear equation. Let's work through the solution process carefully.

We first look for an integrating factor, and we know that one is given by

$$u(t) = e^{-\int 0.05\,dt} = e^{-0.05t}.$$

Hence

$$\left[e^{-0.05t}P\right]' = e^{-0.05t}\left[P' - 0.05P\right] = 2e^{-0.05t}. \tag{3.10}$$

Integrating both sides yields

$$e^{-0.05t}P(t) = -40e^{-0.05t} + C,$$

or

$$P(t) = -40 + Ce^{0.05t}.$$

The initial condition becomes

$$0 = P(0) = -40 + C,$$

so $C = 40$ and the solution is

$$P(t) = 40(e^{0.05t} - 1).$$

To answer the question, we evaluate this when $t = 30$ to get

$$P(30) = 40(e^{1.5} - 1) = 139.2676.$$

Our balance is measured in thousands of dollars, so we learn that our balance after 30 years is \$139,268. ●

Planning for retirement

That's not a bad start for retirement. After all, the amount of money put into savings was \$2000 per year for 30 years, for a total of \$60,000. Because of the compounded interest, this has grown to more than double that amount.

Now, however, let's turn the question around. How much money do you need to retire on?

Example 3.11 After some thought, you have decided that you will need \$50,000 each year to live on after you retire, and that you should plan on living 30 years after your retirement. Assuming that your retirement account will earn 5% interest while you are taking out \$50,000 each year, how much money must be in the retirement account when you retire?

Let $P(t)$ be the balance in your retirement account at time t after retirement. Let P_0 denote the balance in your retirement account when you retire. Then $P(0) = P_0$. The problem is to find P_0 so that $P(30) \geq 0$. Once more we will use a thousand dollars as our unit.

According to the model developed in equation (3.7),

$$P' = 0.05P - 50.$$

The integrating factor we used in (3.10) will also work here. Thus we have

$$\left[e^{-0.05t}P\right]' = -50e^{-0.05t}.$$

Integrating, we get

$$e^{-0.05t}P(t) = 1000e^{-0.05t} + C,$$

or

$$P(t) = 1000 + Ce^{0.05t}.$$

If we let $P(0) = P_0$ denote the balance at the time you retire, then we can evaluate the constant C,

$$P_0 = P(0) = 1000 + C.$$

Therefore, $C = P_0 - 1000$, and the solution is

$$P(t) = 1000 + (P_0 - 1000)e^{0.05t}.$$

Since you want to have that \$50,000 each year until you die 30 years after retiring, you will want $P(30) \geq 0$. If you spend your last cent the day you die, you will want

$$0 = P(30) = 1000 + (P_0 - 1000)e^{1.5}.$$

We can solve this equation for P_0, getting

$$P_0 = 1000(1 - e^{-1.5}) = 776.8698.$$

Thus, you will need to have saved \$776,870 before you retire in order to have the retirement you want. ◉

Saving for retirement

Well, that \$139,000 we saved in Example 3.9 doesn't seem so great any more. You are going to have to do much better than that. But just how are you going to accumulate the funds needed to finance a comfortable retirement?

Example 3.12 After some more thought prompted by the previous example, you decide that you should put a fixed percentage ρ of your salary into your retirement account. The question is, what value of ρ will achieve our goal?

First, you realize that your current salary of \$35,000 per year will not stay at that level forever, hopefully for no more than one year. We need a model of how your salary will grow over time. Let's assume that your salary will grow at 4% per year. That's only a little more than the inflation rate. This thought leads to the differential equation $S' = 0.04S$, where $S(t)$ is your annual salary in thousands of dollars. We are very familiar with the exponential equation, so we easily solve this equation to get

$$S(t) = 35e^{0.04t}.$$

Now you notice that with this model, your salary in year 40 will be over \$173,000. This seems excessive, but you are assured by your financial advisors that this type of increase over a lifetime is not at all unusual. Remember, we are including inflation in our forecasts, and the 4% per year increase we are projecting hardly covers the historical inflation rate. However, there is another concern. If your salary at retirement is this large, what kind of income should you plan for in your retirement years? Clearly the \$50,000 in the previous example is too little. You decide that \$100,000 is a more reasonable figure. This means that the size of your retirement account at retirement has to be double that found in the previous example. (Why?) You decide to be cautious and aim for a retirement fund of \$1,600,000.

We will assume once more that your retirement account will earn an interest rate of 5%. Let $P(t)$ denote the balance in thousands of dollars in your retirement account at time t. The balance P will grow between times t and $t + \Delta t$ from two sources, the interest on the balance, which is $0.05P(t)\Delta t$, and from your investment, which is $\rho\, S(t)\Delta t$. Hence we have

$$P(t + \Delta t) - P(t) \approx 0.05P(t)\Delta t + \rho S(t)\Delta t.$$

Therefore,

$$\begin{aligned} P'(t) &= \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} \\ &= 0.05P(t) + \rho S(t) \\ &= 0.05P(t) + 35\rho e^{0.04t}. \end{aligned}$$

We solve this linear equation as before. Again, the integrating factor $u(t) = e^{-0.05t}$ used in (3.10) will work here. We get

$$\left[e^{-0.05t}P\right]' = 35\rho e^{-0.01t}.$$

Integrating, we get

$$e^{-0.05t}P(t) = 35\rho \int e^{-0.01t}\,dt = -3500\rho e^{-0.01t} + C,$$

or

$$P(t) = -3500\rho e^{0.04t} + Ce^{0.05t}.$$

We will again assume that you are starting with no money in your retirement account, so this initial condition says

$$0 = P(0) = -3500\rho + C,$$

which allows us to conclude that $C = 3500\rho$, and

$$P(t) = 3500\rho\left(e^{0.05t} - e^{0.04t}\right).$$

We can now compute what ρ has to be by comparing this with our goal, which is that $P(40) = 1600$. Completing the computation, you find that you must save 18.77% of your salary to ensure a comfortable retirement. ●

Saving almost 19% of your salary every year seems like a lot. You decide to explore other strategies.

Example 3.13 When your salary is low, at the beginning of your career, saving this much of your salary will be hard. Perhaps even while you are young you should enjoy more of the fruits of your labor. What would happen if you were to start saving at a more modest rate and slowly increase your savings rate over time? You decide that the percent to save in year t is

$$\rho(t) = \frac{Rt}{40}.$$

With this choice, R will be the savings rate just before you retire after 40 years of work. What does R have to be in order to achieve your retirement goal?

The model in this case is similar to what it was in the previous example, but it has to change to accommodate the variability of your savings rate:

$$\begin{aligned} P' &= 0.05P + \rho S \\ &= 0.05P + \frac{Rt}{40} \times 35e^{0.04t} \\ &= 0.05P + 0.875Rte^{0.04t}. \end{aligned}$$

Again this equation is linear, and again the integrating factor $u(t) = e^{-0.05t}$ used in (3.10) will work. Integrating, we get

$$\left[e^{-0.05t}P\right]' = 0.875Rte^{-0.01t},$$

to get

$$\begin{aligned} e^{-0.05t}P(t) &= 0.875R\int te^{-0.01t}\,dt \\ &= -87.5R(t+100)e^{-0.01t} + C, \end{aligned}$$

or

$$P(t) = -87.5R(t+100)e^{0.04t} + Ce^{0.05t}.$$

Using $P(0) = 0$ to compute that $C = 8750R$, we end up with

$$P(t) = 87.5R\left(100e^{0.05t} - (t+100)e^{0.04t}\right).$$

Finally, we use our goal $P(40) = 1600$ to compute that $R = 0.4021$. This means that with this plan, although your savings rate in your early career will be small, in the last year before your retirement you will have to save over 40% of your salary.

Of course other strategies are possible. In particular, you might think about exploring ways to increase the income on your investments. Didn't you hear somewhere that over the long term, stocks return more than 8%? We will explore some other cases and strategies in the exercises.

EXERCISES

1. Suppose that \$1200 is invested at a yearly rate of 5%, compounded continuously.
   (a) Assuming no additional withdrawals or deposits, how much will be in the account after 10 years?
   (b) How long will it take the balance to reach \$5000?
2. Jamal wishes to invest an unknown sum in an account where interest is compounded continuously. Assuming that Jamal makes no additional deposits or withdrawals, what annual interest rate will allow his initial investment to double in exactly five years?
3. Alicia opens an account that pays an annual rate of 6% compounded continuously with an initial investment of \$5000. After that, she deposits an additional \$1200 per year. Assuming that no withdrawals are made and she continues with the same yearly deposit over a period of 10 years, how much will be in the account at the end of the 10-year period?
4. On the day of his birth, Jason's grandmother pledges to make available \$50,000 on his eighteenth birthday for his college education. She negotiates an account paying 6.25% annual interest, compounded continuously, with no initial deposit, but agrees to deposit a fixed amount each year. What annual deposit should be made to reach her goal?
5. Andre inherits \$50,000 from his grandfather's estate. The money is in an account that pays 5% annual interest, compounded continuously. The terms of the inheritance require that \$8000 be withdrawn each year for Andre's educational expenses. If no additional deposits are made, how long will the inheritance last before the funds are completely gone?
6. Clarissa wants to buy a new car. Her loan officer tells her that her annual rate is 8%, compounded continuously, over a four-year term. Clarissa informs her loan officer that she can make equal monthly payments of \$225. How much can Clarissa afford to borrow?
7. David and Mary would like to purchase a new home. They borrow \$100,000 at 8% annual interest, compounded continuously. The term of the loan is 30 years. What fixed, annual payment will satisfy the terms of their loan?
8. Don and Heidi would like to buy a home. They've examined their budget and determined that they can afford monthly payments of \$1000. If the annual interest is 7.25% and the term of the loan is 30 years, what amount can they afford to borrow?
9. José is 25 years old. His current annual salary is \$28,000. Over the next 20 years, he expects his salary to increase continuously at a rate of 1% per year. He establishes a fund paying 6% annual interest, compounded continuously, with an initial deposit of \$2500 and a promise

to deposit a fixed percentage of his annual income each year. Find that fixed percentage if José wants his balance to reach \$50,000 at the end of the 20-year period.

10. Adriana opens a savings account with an initial deposit of \$1000. The annual rate is 6%, compounded continuously. Adriana pledges that each year her annual deposit will exceed that of the previous year by \$500. How much will be in the account at the end of the tenth year?

Discrete versus continuous You may have cast a somewhat skeptical eye at our financial models involving continuous compounding of interest. After all, no one pays off their loans continuously. It would be difficult to imagine how that could be accomplished. Payments are made at regular intervals, perhaps yearly, but more likely monthly. Let's examine how accurately our continuous models reflect the real world of finance.

11. The *recursive* definition,

$$a(n+1) = ra(n), \quad a(0) = a_0,$$

is called a ***first-order difference equation*** and generates the sequence

$$a_0, ra_0, r(ra_0), r(r(ra_0)), \ldots.$$

A little simplification shows that the nth term of this sequence is

$$a(n) = a_0 r^n.$$

Now suppose that I represents the annual interest rate, but the interest is awarded in discrete packets, m times per year. Then the rate awarded during each compounding period is I/m. Consequently, if the initial investment is P_0, the balance is $P_0(1 + I/m)$ at the end of the first compounding period, $P_0(1 + I/m)^2$ at the end of the second compounding period, and so on.

(a) Give a first-order difference equation with an initial condition that generates a sequence describing the balance in the account at the end of each compounding period.

(b) Find a formula for the nth term of the sequence generated by the first-order difference equation created in part (a).

12. Arkady invests \$2000 in an account paying 6% annual interest.

(a) If the interest is compounded continuously and no additional deposits or withdrawals are made, how much will be in the account at the end of 10 years?

(b) If the interest is awarded in discrete annual packets, then you want to use the discrete formula generated in Exercise 11. In this case, $m = 1$, so

$$P(10) = 2000\left(1 + \frac{0.06}{1}\right)^{10} \approx \$3,581.70.$$

Calculate the balance in the account at the end of the 10-year period if the interest is compounded

- semiannually (twice per year)
- monthly (12 times per year)
- daily (365 times per year)

(c) Write a short paragraph explaining the point of this problem.

13. The first-order difference equation

$$a(n+1) = ra(n) + b, \quad a(0) = a_0$$

generates the sequence

$$a_0, ra_0 + b, r(ra_0 + b) + b, r(r(ra_0 + b) + b) + b, \ldots.$$

A little simplification shows that the nth term of this sequence is

$$a(n) = a_0 r^n + b\left(1 + r + r^2 + \cdots + r^{n-1}\right).$$

(a) Show that

$$a(n) = \left(a_0 - \frac{b}{1-r}\right) r^n + \frac{b}{1-r}.$$

(b) Let I represent the annual interest rate, m the number of compounding periods in a year, P_0 the initial investment, and d the fixed deposit at the end of each compounding period. Then the balance at the end of each compounding period is generated by the first order difference equation

$$P(n+1) = \left(1 + \frac{I}{m}\right) P(n) + d, \quad P(0) = P_0.$$

Use the result of part (a) to show that the balance at the end of n compounding periods is given by

$$P(n) = \left(P_0 + \frac{md}{I}\right)\left(1 + \frac{I}{m}\right)^n - \frac{md}{I}.$$

14. Demetrios opens an account with an initial investment of \$2000. The annual interest rate is 5%.

(a) If the interest is compounded continuously and Demetrios makes an additional \$1000 deposit every year, what will be the balance at the end of 10 years?

(b) If the interest is compounded quarterly (four times per year) and \$250 is deposited at the end of each compounding period, what will be the balance after 10 years?

(c) What happens if the interest is compounded daily?

15. Chieh-Hsien purchases a new car and finances \$12,000 at an annual rate of 8% for 5 years.

(a) If the interest is compounded continuously, what are his monthly payments?

(b) If the interest is compounded monthly, what are his monthly payments?

three equilibrium points (not including $\mu = 0$ in your count). Discuss the ecological significance of these regions. Which region leads to a stable, manageable budworm population? Which region leads to a serious outbreak of the budworm population?

6. Pick an (R, Q) in each region determined by the bifurcation curve in Exercise 5 and use your numerical solver to sketch the fate of μ versus τ for various initial population sizes. [Use equation (5.3).]

If you wish to pursue this model in greater depth, here are some resources you will find useful.

- Ludwig, Jones, and Holling. "Qualitative Analysis of Insect Outbreak Systems: The Spruce Budworm and Forest," *Journal of Animal Ecology* (1978), **47**, 315–332.
- Murray, J. *Mathematical Biology* (1993), Springer-Verlag (New York).
- Strogatz, S. *Nonlinear Dynamics and Chaos* (1994), Addison-Wesley (Reading, Mass.).

PROJECT 3.6 Social Security, Now or Later

In the spring of 2000, a change in the Social Security law allowed individuals over their full retirement age[10] to continue to work full time and still receive full social security retirement benefits. However, they could choose to delay the start of their retirement benefits until a later time, in which case the monthly benefit would be larger. In May 2000, all working individuals over full retirement age had an important decision to make: Should I take social security now or later? Anyone who continues to work after their full retirement age will face the same question.

Many people faced with this decision find it an easy choice to make. If the government will give you money, take it. That is not bad logic, but others will want to examine the situation a little more deeply. They ask the question, Which way maximizes my financial situation?

It is not easy to see what this question means, but here is one way to approach the problem. It is based on the idea of setting up an account in which the Social Security benefits are deposited, which pays a certain fixed rate of return and from which taxes are paid on the benefits and on the return. Of course, such an account would be purely hypothetical. No one would actually do this. However, if we postulate two such accounts, one with balance $P_1(t)$ set up at full retirement age and another with balance $P_2(t)$ set up at the true retirement age, say 70, we can compare the balances over time.

Of course, $P_2(t) = 0$ for $t < 70$ while $P_1(t)$ is increasing. Then starting at $t = 70$, P_2 will start increasing at a faster rate than P_1 because of the larger benefits. The question is, At what age will P_2 catch up with P_1? This "catch-up" age might help someone make the decision on when to start receiving Social Security. If the catch-up age is low, then it might be a good idea to delay taking Social Security. On the other hand, if the catch-up age is high, then the opposite strategy might be best. Of course, it is up to each person to say what is high and what is low.

To carry out the modeling of the account balances, it is necessary to accumulate some data.

- The most important data is the monthly benefit amounts for retirement at the full retirement age and 70. The correct figures depend in a fairly complicated way on one's entire income history. However, it is possible to make estimates. Use your Internet browser to visit the Social Security Administration (http://www.ssa.gov/). There you will find calculators that will compute the monthly benefit for you, once you provide it with income data. The point is to find the monthly payment amounts starting at both full retirement age and at 70 based on the same income data. Perhaps the easiest thing to do is to find the maximum benefits for each age. Let A_1 be the annual benefit starting at full retirement age, and A_2 that starting at 70. Furthermore, let T_d be the time in years between full retirement age and 70.
- It will be necessary to deduct the taxes paid on each of the accounts. Consult with IRS documents to discover what portion of the benefits is taxable. In addition, taxes will have to be paid on the income into the accounts. Make an assumption about the marginal tax rate, which we will denote by ρ.
- The only other item of information needed is r, the rate of return on the investment accounts. Since this is uncertain, the computation should be done for a range of return rates varying from 0 to a maximum that will be computed.

With the data you have collected, perform the following tasks.

1. Along the lines of the derivations in Section 3.3, construct a differential equation model for the growth of the two accounts. Assume that all payments are made continuously. Be sure to account for taxes and return on investments.
2. Solve the differential equations exactly.
3. Show that if

$$r > r_{\max} = \frac{1}{(1-\rho)T_d}\ln(A_2/A_1), \tag{6.1}$$

then $P_1(t) > P_2(t)$ for all t. In other words, if the return on investment is large, the account started at the full retirement age is always larger than the one started at 70.

[10] As a result of the same change in the Social Security law, the full retirement age was also changed. It used to be 65 for everyone, but now it varies depending on a person's birth date. For individuals born in 1937 or earlier it is still 65, but for younger people it increases gradually to the point where for people born in 1967 or later it is 67.

4. Show that for $0 < r < r_{max}$ the catch-up time T is

$$T = \frac{1}{(1-\rho)r} \ln\left(\frac{A_2 - A_1}{A_2 e^{-(1-\rho)rT_d} - A_1}\right).$$

Plot the catch-up time T versus the investment rate r with the data you got from the Social Security Administration.

5. Redo the whole exercise with $r = 0$. In other words, assume that there is no return on investment. Show that the catch-up time is

$$T = \frac{T_d A_2}{A_2 - A_1}.$$

6. What would you advise an older relative or friend who asks you if he or she should delay receiving Social Security?