Nonrenewable Resources

Based on the discussion presented in *For all Practical Purposes: Mathematical Literacy in Today's World* (Seventh Edition) edited by COMPAP (2006).

A *nonrenewable resource* is one that does not tend to replenish itself. That is, it is a (natural) resource which cannot be produced, re-grown, regenerated, or reused on a scale which can sustain its consumption rate. Gasoline, coal, and natural gas are important examples, while lottery winnings and inheritances could be examples from personal affairs. There is no practical way to recover or reconstitute these resources after use.

In contrast, resources such as timber (when harvested sustainably) or metals (which can be recycled) are considered *renewable resources*. Some substances, such as aluminum or the sand used to make glass, are potentially recyclable, but to the extent that we do not recycle them, they, too, are nonrenewable.

For a nonrenewable resource, only a fixed supply R is available. Even without human population increases, we face dwindling nonrenewable resources. We are interested in the question: How long will the supply of a resource last?

Constant Consumption. Let $Y(t)$ represent the amount of the resource that has been used by time t. Let $c(t)$ represent the consumption rate in year t. As long as the rate of use of the resource remains constant, the answer is easy. If we are using $c(t) = A$ units per year each year, then

$$\frac{dY}{dt} = A, \quad Y(0) = 0.$$

The solution to this differential equation is $Y(t) = At$. The resource will last $t_R = R/A$ years since $Y(t_r) = R$. R/A is called the *static reserve* (or *static index*) and it denotes the amount of time a nonrenewable resource of size R will last if used at a constant rate A. We will let $S = R/A$

This kind of calculation is the basis for statements such as those claiming that at the current rate of consumption, U.S. recoverable coal reserves will last 250 years or that the U.S. strategic reserve of gasoline (stored in underground salt domes in the South) will last 60 days.

However, the rate of use of resources tends to increase with population and with a higher standard of living. For example, projections for use of electric power often assume that use will increase by a fixed percentage each year. This is the simplest situation (apart from constant usage) and one that we can easily model.

Non-constant Consumption: Fixed Percentage Increase. Suppose the initial consumption rate is $c(0) = A$ and each year, the consumption rate increases by $(r \times 100)\%$.

Then

$$\frac{dY}{dt} = A(1+r)^t, \quad Y(0) = 0.$$

This is a separable first order ODE:

$$\begin{aligned}
dY &= A(1+r)^t dt \\
\int dY &= \int A e^{t\ln(1+r)} dt \\
Y &= \frac{A}{\ln(1+r)} e^{t\ln(1+r)} + C \\
&= \frac{A}{\ln(1+r)} (1+r)^t + C \\
Y(0) &= \frac{A}{\ln(1+r)} + C \Longrightarrow C = -\frac{A}{\ln(1+r)} \\
Y(t) &= \frac{A}{\ln(1+r)}((1+r)^t - 1)
\end{aligned}$$

When is the resource used up? We need to find t_R such that $Y(t_R) = R$:

$$\begin{aligned}
\frac{A}{\ln(1+r)}((1+r)^t - 1) &= R \\
(1+r)^t - 1 &= \frac{R\ln(1+r)}{A} \\
(1+r)^t &= \frac{R\ln(1+r)}{A} + 1 \\
t_r &= \log_{(1+r)}\left(\frac{R\ln(1+r)}{A} + 1\right) \\
&= \log_{(1+r)}[S\ln(1+r) + 1] \\
&= \frac{\ln[S\ln(1+r) + 1]}{\ln(1+r)}
\end{aligned}$$

When r is small (say $0 < r \leq .1$), $\ln(1+r) \approx r$ so

$$t_r \approx \frac{\ln(Sr+1)}{\ln(1+r)}$$

Non-constant Consumption: Exponential Increase. Suppose the initial consumption rate is $c(0) = A$ and each year, the consumption rate increases exponentially. Then

$$\frac{dY}{dt} = Ae^{kt}, \quad Y(0) = 0.$$

This is a separable first order ODE:

$$\begin{aligned} dY &= Ae^{kt}dt \\ Y &= \frac{A}{k}e^{kt} + C \\ Y(0) &= \frac{A}{k} + C \Longrightarrow C = -\frac{A}{k} \\ Y(t) &= \frac{A}{k}(e^{kt} - 1) \end{aligned}$$

When is the resource used up? We need to find t_R such that $Y(t_R) = R$:

$$\begin{aligned} \frac{A}{k}(e^{kt} - 1) &= R \\ e^{kt} - 1 &= \frac{Rk}{A} \\ e^{kt} &= \frac{Rk}{A} + 1 \\ kt &= \ln\left(\frac{Rk}{A} + 1\right) \\ t_R &= \frac{1}{k}\ln\left(\frac{Rk}{A} + 1\right) \\ &= \frac{1}{k}\ln(Sk + 1) \end{aligned}$$

The time it takes for a nonrenewable resource to be exhausted if consumed at an exponential rate is called the *exponential reserve* (or *exponential index*). Observe that this formula is quite similar to the one obtained when there was a fixed percentage increase. (Just let $k = \ln(1 + r)$.) So when r is small, we can use the exponential increase model as a fairly good approximation to the fixed percentage increase model. We will use this approximation for the remainder of this document unless stated otherwise.

Example: US Coal Reserves. The static reserve for US coal is 250 years. If the rate of consumption increases 2.25% per year, what is the exponential reserve (index)?

ANSWER: We can approximate $k = \ln 1 + r \approx r$ so

$$t_R = \frac{1}{.0225}\ln((250)(.0225) + 1) = 84 \text{ years}$$

We must not take such projections as exact predictions. Estimates of supplies of a resource may underestimate how much is available, and previously unknown sources may be discovered or the technology improved to extract previously unavailable supplies. In addition, as supplies dwindle, the economic considerations of supply, demand, and price come into play. We will never completely run out of oil. It will always be available "at a price."

We must not take such projections lightly, either, because we are discussing resources that, once used, are gone forever. In any projection, it is very important to examine the assumptions, because small differences in the rate of increase of use can make big differences in the exponential reserve.

Example: Using Up Retirement Savings. Suppose that you begin retirement with $1 million in savings, and you do not trust banks or the stock market, so you keep it all under your mattress. Suppose that it costs you $50,000 per year to live at your accustomed standard of living and there is no inflation. How long will your retirement nest egg last? What is inflation is 5% per year?

ANSWER: The static reserve (index) is $S = 1,000,000/50,000 = 20$ years. If, however, there is constant 5% per year inflation, then it will cost you increasingly more per year to live, so you should realize that your savings will last only for the length of the exponential reserve, which is

$$t_R = \frac{1}{.05} \ln\left((20)(.05) + 1\right) = 14 \text{ years}$$

You have a fine strategy if you expect to live just 14 more years and want to die broke!

In our examples so far, we have assumed that the resource is just sitting there, waiting to be used up. For many natural resources, however, we have to find and develop new sources. As doing that becomes more difficult and more costly, at some point the exponentially increasing demand outstrips the ability to meet that demand.

Natural resources such as coal, petroleum, oil and natural gas take thousands of years to form naturally and cannot be replaced as fast as they are being consumed. Solar, wind, wave, and geothermal energies are based on renewable resources. Renewable resources such as the movement of water (hydropower, including tidal power; ocean surface waves used for wave power), wind (used for wind power), geothermal heat (used for geothermal power); and radiant energy (used for solar power) are practically infinite and cannot be depleted, unlike their non-renewable counterparts, which are likely to run out if not used wisely. Still, these technologies are not fully utilized.

Example. By the time there is concern about using up a nonrenewable resource, it may be too late. Suppose that a resource has a static reserve (index) of 1000 years, but consumption is growing at 2% per year. (a) How long will the resource last? (b) How long before half the resource is gone? (c) How much longer will the resource last if, after half of it is gone, consumption is stabilized at the then-current level? (d) What implications do you see to your answers?

ANSWERS:

(a)

$$t_R = \frac{1}{.02} \ln\left((1000)(.02) + 1\right) = 152 \text{ years.}$$

(b) We need to find t such that

$$Y(t) = \frac{A}{.02}(e^{0.02t} - 1) = 0.5R$$

and we know that $S = R/A = 1000$. Simplifying

$$\begin{aligned} e^{0.02t} - 1 &= 0.01S = 10 \\ e^{0.02t} &= 11 \\ 0.02t &= \ln 11 \\ t &= \frac{\ln 11}{0.02} = 120 \text{ years.} \end{aligned}$$

(c) The consumption rate is $c(t) = A(1.02)^t \approx Ae^{0.02t}$. In year 120, the consumption rate is fixed at the constant $B = Ae^{0.02(120)} = Ae^{2.4}$. The amount of time until the remaining resource is consumed is

$$\frac{0.5R}{Ae^{2.4}} = \frac{0.5}{e^{2.4}}S = 45 \text{ additional years.}$$

Thus the total time the resource would be available is $120 + 45 = 165$ years. We would gain 165 - 152 = 13 years.

(d) While 13 additional years may not seem like a lot, that might be enough time for research to either identify an alternative resource (with a larger reserve) or develop a more efficient process that decreases the consumption rate further. But it would be better not to wait until a resource is half used up to start thinking about alternatives!