More Nonlinear Programming Examples

Example 1

A manufacturing firm produces two products, A and B, using two limited resources. The maximum amounts of resources 1 and 2 available per day are 1000 and 250 units respectively. The production of 1 unit of product A requires 1 unit of resource 1 and 0.2 units of resource 2. The production of 1 unit of product B requires 0.5 units of resource 1 and 0.5 units of resource 2. The unit costs of resources 1 and 2 are given by the relations $(0.375 - 0.00005u_1)$ and $(0.75 - 0.0001u_2)$, respectively, where u_i denotes the number of units of resource i used. The selling prices per unit of products A and B, p_A and p_B, are given by

$$p_A = 2.00 - 0.0005x_A - 0.00015x_B$$
$$p_B = 3.50 - 0.0002x_A - 0.0015x_B$$

where x_A and x_B indicate, respectively, the number of units of products A and B sold. Formulate the problem of maximizing the profit assuming that the firm can sell all the units it manufactures.

max profit = revenue − cost

$$= x_A(2 - 0.0005x_A - 0.00015x_B)$$
$$+ x_B(3.5 - 0.0002x_A - 0.0015x_B)$$
$$- (x_A + 0.5x_B)[0.375 - 0.00005(x_A + 0.5x_B)]$$
$$- (0.2x_A + 0.5x_B)[0.75 - 0.0001(0.2x_A + 0.5x_B)]$$

s.t. $x_A + 0.5x_B \le 1000$ (resource 1)

$0.2x_A + 0.5x_B \le 250$ (resource 2)

$x_A, x_B \ge 0$

Example 2

A manufacturer of a particular product produces x_1 units in the first week and x_2 units in the second. The number of units produced in the first and second weeks must be at least 200 and 400 respectively, to be able to supply the regular customers. The initial inventory is zero and the manufacturer ceases to produce the product at the end of week 2. The production cost of a unit, in dollars, is given by $4x_i^2$ where x_i is the number of units produced in week i. In addition to the production cost, there is an inventory cost of \$10 per unit for each unit produced in the first week that is not sold by the end of the first week. Formulate the problem of minimizing the total cost.

In addition to the x_j, define S_1 = surplus production in week 1. Assume we end wk 2 with no inventory.

$$\min \; 4x_1^3 + 4x_2^3 + 10S_1$$

$$\text{s.t.} \quad x_1 - S_1 = 200 \quad (\text{wk 1 demand})$$

$$S_1 + x_2 = 400 \quad (\text{wk 2 demand})$$

$$x_1, x_2, S_1 \geq 0 \quad (\text{and integer})$$

Since $x_1 = 200 + S_1$, $x_2 = 400 - S_1$

can write $\min f(S_1) = 4(200+S_1)^3 + 4(400-S_1)^3 + 10S_1$

s.t. $S_1 \in [200, 400]$ and integer

$$\frac{df}{dS_1} = 12(200+S_1) - 12(400-S_1) + 10$$

$$\frac{d^2f}{dS_1^2} = 12(-200) < 0 \Rightarrow f \text{ is concave max!}$$

$$\frac{df}{dS_1} = 0 \text{ when } S_1 = \frac{2390}{24} \approx 99.5$$

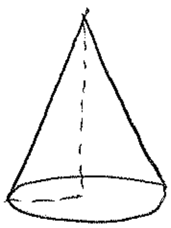
If $S_1 = 99$ $f(S_1) = 216{,}008{,}190$

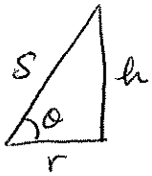
$S_1 = 100$ $f(S_1) = 216{,}000{,}100$ $\Rightarrow$

$$\boxed{\begin{array}{c} S_1^* = 100 \\ x_1^* = x_2^* = 300 \end{array}}$$

Example 3

An electric light is placed directly over the center of a circular plot of lawn 100 m in diameter. Assuming that the intensity of the light varies directly as the sine of the angle at which it strikes an illuminated surface, and inversely as the square of its distance from the surface, how high should the light be hung in order that the intensity may be as great as possible at the circumference of the plot?





S = distance to light $= \sqrt{h^2 + r^2}$

$\text{sen}\ \theta = \dfrac{h}{S}$

so intensity $I \propto \dfrac{\text{sen}\ \theta}{S^2} \Leftrightarrow I = k \dfrac{h}{(h^2+r^2)^{3/2}}$

where k is constant

$$\frac{dI}{dh} = k\left[h(-3/2)(h^2+r^2)^{-5/2}(2h) + (h^2+r^2)^{-3/2}(1)\right]$$

$$\frac{dI}{dh} = 0 \text{ when } \frac{-3h^2+h^2+r^2}{(h^2+r^2)^{5/2}} = 0 \Rightarrow h^* = \sqrt{\frac{r^2}{2}}$$

$$\frac{d^2I}{dh^2} = \frac{-15h^3 - 9h(h^2+r^2)}{(h^2+r^2)^{7/2}} < 0 \text{ for all } h > 0$$

since the numerator $-24h^3 - 9hr^2 < 0$

and the denominator is always nonneg

$\Rightarrow$ we have a local max!

For this specific problem $h^* = \sqrt{\dfrac{50^2}{2}} \approx 35{,}36\ m$