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Chapter 9

Multivariable Calculus

9.1 FUNCTIONS OF SEVERAL VARIABLES

In previous chapters we dealt with functions of a single independent variable x, which were generally expressed as $y = f(x)$. Since many economic activities involve more than one independent variable, we now turn our attention to functions of two or more independent variables. $z = f(x, y)$ is defined as a *function of two independent variables* if there exists one and only one value of z in the range of f for each ordered pair of real numbers (x, y) in the domain of f. By convention, z is the *dependent variable*; x and y, the *independent variables*.

EXAMPLE 1. If a firm produces one good x for which the cost function is $C(x) = 350 + 8x$ and another good y for which the cost function is $C(y) = 225 + 6y$, the total cost to the firm can be expressed simply as

$$C(x, y) = 350 + 8x + 225 + 6y = 575 + 8x + 6y$$

Other examples of multivariable functions include

$$f(x, y) = x^2 + 2xy + y^2$$
$$z(x, y) = 4xy^2$$

EXAMPLE 2. Multivariable functions can be evaluated for specific values of x and y, such as $x = 3$, $y = 5$, by replacing x and y with the desired values. Using the functions from Example 1,

$$C(3, 5) = 575 + 8(3) + 6(5) = 629$$
$$f(3, 5) = (3)^2 + 2(3)(5) + (5)^2 = 64$$
$$z(3, 5) = 4(3)(5)^2 = 300$$

9.2 PARTIAL DERIVATIVES

Given a multivariable function $z = f(x, y)$, the *partial derivative* measures the rate of change of the dependent variable (z) with respect to one of the independent variables (x or y) while the other independent variable (y or x) is held constant. There is consequently a distinct partial derivative for each of the independent variables.

The *partial derivative of z with respect to x* measures the rate of change of z with respect to x while y is held constant. It is written $\partial z/\partial x$, $\partial f/\partial x$, $f_x(x, y)$, f_x, or z_x, and is defined as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{9.1a}$$

Similarly, the *partial derivative of z with respect to y* measures the rate of change of z with respect to y while x is held constant. It is written $\partial z/\partial y$, $\partial f/\partial y$, $f_y(x, y)$, f_y, or z_y, and is defined by

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{9.1b}$$

To find the partial derivative of a function with respect to one of the independent variables, simply treat the other independent variable as a constant and follow the ordinary rules of differentiation. Illustrations are found in Examples 3, 4, and 5, and Problems 9.1–9.8; verification, in Problem 9.28.

EXAMPLE 3. The partial derivatives of a multivariable function such as $z = 5x^3y^4$ are found as follows:

(*a*) When differentiating with respect to x, treat the y term as a constant by mentally joining it with the coefficient:

$$z = (5y^4) \cdot x^3$$

then take the derivative of the x term, holding the y term constant,

$$\frac{\partial z}{\partial x} = (5y^4) \cdot \frac{d}{dx}(x^3) = (5y^4) \cdot 3x^2$$

Recalling that a multiplicative constant remains in the process of differentiation, now simply multiply and rearrange terms to obtain

$$\frac{\partial z}{\partial x} = 15x^2y^4$$

(*b*) When differentiating with respect to y, treat the x term as a constant by considering it part of the coefficient; then take the derivative as we did above:

$$z = (5x^3) \cdot y^4$$

$$\frac{\partial z}{\partial y} = (5x^3) \cdot \frac{d}{dy}(y^4) = (5x^3) \cdot 4y^3 = 20x^3y^3$$

EXAMPLE 4. To find the partial derivatives for $z = 6x^3 - 7x^2y^2 + 4y^5$:

(*a*) When differentiating with respect to x, mentally bracket off all y terms to remember to treat them as constants:

$$z = 6x^3 - (7y^2)x^2 + (4y^5)$$

then take the derivative of each term, recalling that while multiplicative constants remain in the differentiation process, additive constants drop out because the derivative of a constant is zero.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{d}{dx}(6x^3) - (7y^2) \cdot \frac{d}{dx}(x^2) + \frac{d}{dx}(4y^5)\\ &= 18x^2 - (7y^2) \cdot 2x + 0\\ &= 18x^2 - 14xy^2\end{aligned}$$

(*b*) When differentiating with respect to y, block off all the x terms and then differentiate as above.

$$z = (6x^3) - (7x^2)y^2 + 4y^5$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{d}{dy}(6x^3) - (7x^2) \cdot \frac{d}{dy}(y^2) + \frac{d}{dy}(4y^5)\\ &= 0 - (7x^2) \cdot 2y + 20y^4\\ &= -14x^2y + 20y^4\end{aligned}$$

9.3 RULES OF PARTIAL DIFFERENTIATION

Partial derivatives follow the same basic patterns as the rules of differentiation in Section 5.2. A few key rules are given below, demonstrated in Examples 5–9, and treated in Problems 9.2–9.8

9.3.1 *Product Rule*

Given $z = g(x, y) \cdot h(x, y)$,

$$\frac{\partial z}{\partial x} = g(x, y) \cdot \frac{\partial h}{\partial x} + h(x, y) \cdot \frac{\partial g}{\partial x} \qquad (9.2a)$$

$$\frac{\partial z}{\partial y} = g(x, y) \cdot \frac{\partial h}{\partial y} + h(x, y) \cdot \frac{\partial g}{\partial y} \qquad (9.2b)$$

EXAMPLE 5. Given $z = (7x + 2y)(5x + 9)$, by the product rule,

$$\frac{\partial z}{\partial x} = (7x + 2y)(5) + (5x + 9)(7) = 70x + 63 + 10y$$

$$\frac{\partial z}{\partial y} = (7x + 2y)(0) + (5x + 9)(2) = 10x + 18$$

9.3.2 Quotient Rule

Given $z = g(x, y)/h(x, y)$, $h(x, y) \neq 0$,

$$\frac{\partial z}{\partial x} = \frac{h(x, y) \cdot (\partial g/\partial x) - g(x, y) \cdot (\partial h/\partial x)}{[h(x, y)]^2} \tag{9.3a}$$

$$\frac{\partial z}{\partial y} = \frac{h(x, y) \cdot (\partial g/\partial y) - g(x, y) \cdot (\partial h/\partial y)}{[h(x, y)]^2} \tag{9.3b}$$

EXAMPLE 6. Given $z = (3x + 8y)/(4x + 7y)$, by the quotient rule,

$$\frac{\partial z}{\partial x} = \frac{(4x + 7y)(3) - (3x + 8y)(4)}{(4x + 7y)^2}$$

$$= \frac{12x + 21y - 12x - 32y}{(4x + 7y)^2} = \frac{-11y}{(4x + 7y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(4x + 7y)(8) - (3x + 8y)(7)}{(4x + 7y)^2}$$

$$= \frac{32x + 56y - 21x - 56y}{(4x + 7y)^2} = \frac{11x}{(4x + 7y)^2}$$

9.3.3 Generalized Power Function Rule

Given $z = [g(x, y)]^n$,

$$\frac{\partial z}{\partial x} = n[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial x} \tag{9.4a}$$

$$\frac{\partial z}{\partial y} = n[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial y} \tag{9.4b}$$

EXAMPLE 7. Given $z = (x^4 + 5y^2)^3$, by the generalized power function rule,

$$\frac{\partial z}{\partial x} = 3(x^4 + 5y^2)^2 \cdot (4x^3) = 12x^3(x^4 + 5y^2)^2$$

$$\frac{\partial z}{\partial y} = 3(x^4 + 5y^2)^2 \cdot (10y) = 30y(x^4 + 5y^2)^2$$

9.3.4 Natural Exponential Function Rule

Given $z = e^{g(x, y)}$

$$\frac{\partial z}{\partial x} = e^{g(x, y)} \cdot \frac{\partial g}{\partial x} \tag{9.5a}$$

$$\frac{\partial z}{\partial y} = e^{g(x, y)} \cdot \frac{\partial g}{\partial y} \tag{9.5b}$$

EXAMPLE 8. Given $z = e^{5xy^2}$, by the natural exponential function rule,

$$\frac{\partial z}{\partial x} = e^{5xy^2} \cdot 5y^2 = 5y^2e^{5xy^2} \qquad \frac{\partial z}{\partial y} = e^{5xy^2} \cdot 10xy = 10xye^{5xy^2}$$

9.3.5 *Natural Logarithmic Function Rule*

Given $z = \ln |g(x, y)|$,

$$\frac{\partial z}{\partial x} = \frac{1}{g(x, y)} \cdot \frac{\partial g}{\partial x} \tag{9.6a}$$

$$\frac{\partial z}{\partial y} = \frac{1}{g(x, y)} \cdot \frac{\partial g}{\partial y} \tag{9.6b}$$

EXAMPLE 9. Given $z = \ln |4x + y^2|$, by the natural logarithmic function rule,

$$\frac{\partial z}{\partial x} = \frac{1}{4x + y^2} \cdot 4 = \frac{4}{4x + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{4x + y^2} \cdot 2y = \frac{2y}{4x + y^2}$$

9.4 SECOND-ORDER PARTIAL DERIVATIVES

Given a function $z = f(x, y)$, *the second-order partial derivative* signifies that the function has been differentiated partially with respect to one of the independent variables twice while the other independent variable has been held constant:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \qquad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

In words, f_{xx} measures the rate of change of the first-order partial derivative f_x with respect to x while y is held constant. f_{yy} is exactly parallel. See Problems 9.9–9.10.

The *cross* (*or mixed*) *partial derivative*, f_{xy} or f_{yx}, indicates that the primitive function has been first partially differentiated with respect to one independent variable and that the resulting partial derivative has in turn been partially differentiated with respect to the other independent variable:

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y}$$

In brief, a cross partial measures the rate of change of a first-order partial derivative with respect to the other independent variable. Note how the order of independent variables differs in the different forms of notation. See Problems 9.11–9.12.

EXAMPLE 10. The (*a*) first, (*b*) second, and (*c*) cross partial derivatives for $z = 6x^4 + 5xy + 3y^6$ are taken as shown below.

(*a*)
$$\frac{\partial z}{\partial x} = z_x = 24x^3 + 5y \qquad \frac{\partial z}{\partial y} = z_y = 5x + 18y^5$$

(*b*)
$$\frac{\partial^2 z}{\partial x^2} = z_{xx} = 72x^2 \qquad \frac{\partial^2 z}{\partial y^2} = z_{yy} = 90y^4$$

(*c*)
$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(24x^3 + 5y) = z_{xy} = 5$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(5x + 18y^5) = z_{yx} = 5$$

EXAMPLE 11. The (*a*) first, (*b*) second, and (*c*) cross partial derivatives for $z = 5x^2y^3$ are evaluated below at $x = 4$, $y = 1$.

(*a*)
$$z_x = 10xy^3 \qquad z_y = 15x^2y^2$$
$$z_x(4, 1) = 10(4)(1)^3 = 40 \qquad z_y(4, 1) = 15(4)^2(1)^2 = 240$$

(*b*)
$$z_{xx} = 10y^3 \qquad z_{yy} = 30x^2y$$
$$z_{xx}(4, 1) = 10(1)^3 = 10 \qquad z_{yy}(4, 1) = 30(4)^2(1) = 480$$

(*c*)
$$z_{xy} = \frac{\partial}{\partial y}(10xy^3) = 30xy^2 \qquad z_{yx} = \frac{\partial}{\partial x}(15x^2y^2) = 30xy^2$$
$$z_{xy}(4, 1) = 30(4)(1)^2 = 120 \qquad z_{yx}(4, 1) = 30(4)(1)^2 = 120$$

By *Young's theorem*, if both cross partial derivatives are continuous, they will be identical. See Problems 9.11–9.12.

9.5 OPTIMIZATION OF MULTIVARIABLE FUNCTIONS

For a multivariable function such as $z = f(x, y)$ in Fig. 9-1 to be at a relative minimum or maximum, three conditions must be met:

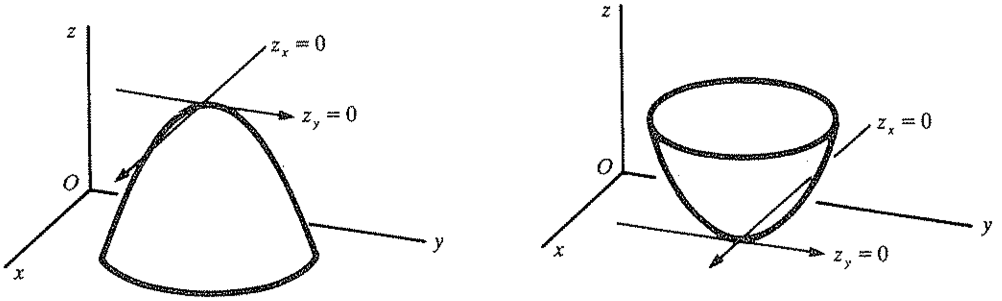
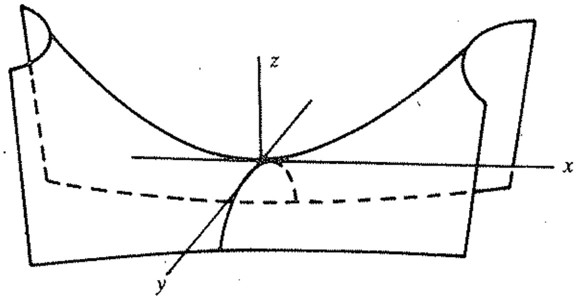


Fig. 9-1

1. The first-order partial derivatives must equal zero simultaneously: $f_x(a, b) = f_y(a, b) = 0$. This indicates that at the given point (a, b), called a *critical point*, the function is neither increasing nor decreasing but is at a relative plateau with respect to the principal axes.
2. The second-order direct partial derivatives, when evaluated at the critical point (a, b), must both be positive for a minimum $[f_{xx}(a, b),\ f_{yy}(a, b) > 0]$ and negative for a maximum $[f_{xx}(a, b),\ f_{yy}(a, b) < 0]$. This ensures that from a relative plateau at the critical point the function is moving upward in relation to the principal axes in the case of a minimum, and downward in relation to the principal axes in the case of a maximum.
3. The product of the second-order direct partials evaluated at the critical point must exceed the product of the cross partials also evaluated at the critical point. Since $f_{xy} = f_{yx}$, this is written $f_{xx}(a, b) \cdot f_{yy}(a, b) > [f_{xy}(a, b)]^2$ or $f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$.

   Note: If $f_{xx} \cdot f_{yy} < (f_{xy})^2$, (*a*) when f_{xx} and f_{yy} have the same sign, the function is at an *inflection point*; (*b*) when f_{xx} and f_{yy} have different signs, the function is at a *saddle point*, as is illustrated in Fig. 9-2, where the function is at a maximum when viewed from one axis but at a minimum when viewed from the other axis. If $f_{xx} \cdot f_{yy} = (f_{xy})^2$, the test is inconclusive.

For applications, see Example 12 and Problems 9.13, 9.17–9.19.



Saddle Point

Fig. 9-2

EXAMPLE 12. Find the critical points and determine whether the function is at a relative minimum or maximum, given

$$f(x, y) = 2x^3 - y^3 - 24x + 75y + 7$$

(*a*) Take the first-order partial derivatives, set them equal to zero, and solve for x and y:

$$\begin{aligned} f_x &= 6x^2 - 24 = 0 & \qquad f_y &= -3y^2 + 75 = 0 \\ x^2 &= 4 & y^2 &= 25 \\ x &= \pm 2 & y &= \pm 5 \end{aligned} \tag{9.7}$$

With $x = \pm 2$, $y = \pm 5$, there are four distinct sets of critical points: $(2, 5)$, $(2, -5)$, $(-2, 5)$, and $(-2, -5)$.

(*b*) From (*9.7*), take the second-order direct partials, evaluate them at each of the critical points, and check the signs:

$$\begin{array}{lcc} & f_{xx} = 12x & f_{yy} = -6y \\ (1) & f_{xx}(2, 5) = 12(2) = 24 > 0 & f_{yy}(2, 5) = -6(5) = -30 < 0 \\ (2) & f_{xx}(2, -5) = 12(2) = 24 > 0 & f_{yy}(2, -5) = -6(-5) = 30 > 0 \\ (3) & f_{xx}(-2, 5) = 12(-2) = -24 < 0 & f_{yy}(-2, 5) = -6(5) = -30 < 0 \\ (4) & f_{xx}(-2, -5) = 12(-2) = -24 < 0 & f_{yy}(-2, -5) = -6(-5) = 30 > 0 \end{array}$$

With different signs for each of the second partials in (1) and (4), the function cannot possibly be at a relative minimum or maximum at $(2, 5)$ or $(-2, -5)$; it is at a saddle point.

With both signs positive in (2) and negative in (3), the function *may be* at a relative minimum at $(2, -5)$ and at a relative maximum at $(-2, 5)$, but the third condition must be tested first to ensure against the possibility of an inflection point. See Problem 9.13(c).

(*c*) From (*9.7*), take the cross partial derivatives and check to make sure that $f_{xx}(a, b) \cdot f_{yy}(a, b) > [f_{xy}(a, b)]^2$.

$$f_{xy} = 0 \qquad f_{yx} = 0$$

$$f_{xx}(a, b) \cdot f_{yy}(a, b) > [f_{xy}(a, b)]^2$$

From (2),

$$(24) \cdot (30) > (0)^2$$
$$720 > 0$$

From (3),

$$(-24) \cdot (-30) > (0)^2$$
$$720 > 0$$

We can now be sure that the function is indeed at a relative minimum at $(2, -5)$ and at a relative maximum at $(-2, 5)$. See Problems 9.13 and 9.17–9.19. For instances of saddle points, see Problem 9.13(*b*) and (*d*); for inflection points, see Problem 9.13(*c*).

9.6 CONSTRAINED OPTIMIZATION AND LAGRANGE MULTIPLIERS

Many business and science problems call for optimizing a function subject to a given constraint. Given a function $f(x, y)$ subject to a constraint $g(x, y) = k$ (a constant), a helpful new function can be formed simply by (1) setting the constraint equal to zero, (2) multiplying it by λ (the *Lagrange multiplier*), and (3) adding the product to the original function to obtain

$$F(x, y, \lambda) = f(x, y) + \lambda[g(x, y) - k] \tag{9.8}$$

Here $F(x, y, \lambda)$ is the *Lagrangian function*, $f(x, y)$ is the original or *objective function*, and $g(x, y)$ is the *constraint*. Since the constraint is always set equal to zero, the product $\lambda[g(x, y) - k]$ also equals zero and the addition of the term does not change the value of the objective function. Critical values x_0, y_0, and λ_0, at which the function is optimized, are found by taking the partial derivatives of F with respect to *all three* independent variables, setting them equal to zero, and solving simultaneously:

$$F_x(x, y, \lambda) = 0 \qquad F_y(x, y, \lambda) = 0 \qquad F_\lambda(x, y, \lambda) = 0$$

See Example 13 and Problems 9.14, 9.20–9.27.

EXAMPLE 13. To minimize the function

$$f(x, y) = 3x^2 + 5xy + 8y^2$$

subject to the constraint $x + y = 48$,

(*a*) Set the constraint equal to zero,

$$x + y - 48 = 0 \qquad \text{or} \qquad 48 - x - y = 0$$

Multiply it by λ and add it to the objective function to form the Lagrangian function F.

$$F = 3x^2 + 5xy + 8y^2 + \lambda(x + y - 48) \tag{9.9}$$

(*b*) Take the first-order partials, set them equal to zero, and solve simultaneously.

$$F_x = 6x + 5y + \lambda = 0 \tag{9.10}$$

$$F_y = 5x + 16y + \lambda = 0 \tag{9.11}$$

$$F_\lambda = x + y - 48 = 0 \tag{9.12}$$

Subtracting (*9.11*) from (*9.10*) gives

$$x - 11y = 0 \qquad x = 11y$$

Substituting $x = 11y$ in (*9.12*),

$$11y + y = 48 \qquad y_0 = 4$$

from which we find

$$x_0 = 44 \qquad \lambda_0 = -284$$

Substituting the critical values in (*9.9*),

$$F = 3(44)^2 + 5(44)(4) + 8(4)^2 + (-284)(44 + 4 - 48)$$
$$= 3(1936) + 5(176) + 8(16) - 284(0) = 6816$$

Note $F = f$ at the critical values. See Problems 9.14, 9.20–9.27.

9.7 TOTAL DIFFERENTIALS

The *total differential* measures the change in the dependent variable brought about by a small change in each of the independent variables. Given $z = f(x, y)$, the total differential (dz) is expressed mathematically as

$$dz = z_x \cdot dx + z_y \cdot dy \tag{9.13}$$

where z_x and z_y are the partial derivatives of z with respect to x and y, respectively, and dx and dy represent small changes in x and y. Recalling that a derivative (z_x, z_y) measures a *rate* of change while a differential (dx, dy) measures merely a change, the definition in (*9.13*) simply states that a change in z (dz) equals the rate at which z changes for a small change in x (z_x) times some small change in x (dx) plus the rate at which z changes for a small change in y (z_y) times some small change in y (dy). See Example 14 and Problem 9.15.

EXAMPLE 14. Given $z = x^3 + 7xy + 5y^4$, the total differential dz is found by taking the first-order partial derivatives:

$$z_x = 3x^2 + 7y \qquad z_y = 7x + 20y^3$$

and substituting them in the formula presented in (*9.13*),

$$dz = (3x^2 + 7y)\,dx + (7x + 20y^3)\,dy$$

9.8 PRACTICAL APPLICATIONS

A firm faces a Cobb-Douglas production function given by

$$q = K^{2/3}L^{1/3}$$

where q = units of output, K = units of capital, and L = units of labor. The firm has \$180 to spend on production; the price of capital is \$6 per unit, the price of labor is \$2 per unit. To find the maximum level of output the firm can produce subject to its budgetary constraint, using Lagrange multipliers:

1. Set the constraint equal to zero, multiply it by λ, and add it to the objective function (q) to form the Lagrangian function (Q).

$$Q = K^{2/3}L^{1/3} + \lambda(6K + 2L - 180)$$

2. Take all the first-order partial derivatives, set them equal to zero, and solve simultaneously for x_0, y_0 (and λ_0 if desired).

$$\frac{\partial Q}{\partial K} = Q_K = \frac{2}{3}\,K^{-(1/3)}L^{1/3} + 6\lambda = 0 \tag{9.14}$$

$$\frac{\partial Q}{\partial L} = Q_L = \frac{1}{3}\,K^{2/3}L^{-(2/3)} + 2\lambda = 0 \tag{9.15}$$

$$\frac{\partial Q}{\partial \lambda} = Q_\lambda = 6K + 2L - 180 = 0 \tag{9.16}$$

From (*9.14*),

$$-\lambda = \tfrac{1}{9}K^{-(1/3)}L^{1/3} \tag{9.17}$$

From (*9.15*),

$$-\lambda = \tfrac{1}{6}K^{2/3}L^{-(2/3)} \tag{9.18}$$

Equating (*9.17*) and (*9.18*),

$$\tfrac{1}{9}K^{-(1/3)}L^{1/3} = \tfrac{1}{6}K^{2/3}L^{-(2/3)}$$

Recalling from Section 1.3 that exponents are added in multiplication, to solve for L in terms of K, multiply both sides of the equation by the following product: (K raised to the exponent which when added to $-\frac{1}{3}$ gives 0) · (L raised to the exponent which when added to $\frac{1}{3}$ gives 1), namely, $K^{1/3}L^{2/3}$.

$$(K^{1/3}L^{2/3})\cdot\tfrac{1}{9}K^{-(1/3)}L^{1/3} = \tfrac{1}{6}K^{2/3}L^{-(2/3)}\cdot(K^{1/3}L^{2/3})$$

$$\tfrac{1}{9}L = \tfrac{1}{6}K \qquad L = 1.5K$$

Substituting in (*9.16*),

$$6K + 2(1.5K) = 180 \qquad K_0 = 20$$

Substituting $K_0 = 20$ in (*9.16*),

$$L_0 = 30$$

See Problems 9.16–9.27.

Solved Problems

FIRST-ORDER PARTIAL DERIVATIVES

9.1 Find the first-order partial derivatives for each of the following functions. Practice the use of different notations and note how the principles learned earlier are extended to functions of more than two independent variables.

(*a*) $f(x, y) = 11x^4y^7$ (*b*) $f(x, y) = 4x^3 - 8xy - 7y^4$
(*c*) $z = 5x^6 + 12x^2y - 3y^5$ (*d*) $f(x, y, z) = 8x^2y^4z^5$
(*e*) $z = 9w^2 - 4wx + 6x^5 - 3xy + 7y^2$ (*f*) $z = 13w^3 + 3w^2x^3y^4 - 10x^4 - 11y^3$

(*a*)
$$f_x(x, y) = 44x^3y^7 \qquad f_y(x, y) = 77x^4y^6$$

(*b*)
$$\frac{\partial f}{\partial x} = 12x^2 - 8y \qquad \frac{\partial f}{\partial y} = -8x - 28y^3$$

(*c*)
$$\frac{\partial z}{\partial x} = 30x^5 + 24xy \qquad \frac{\partial z}{\partial y} = 12x^2 - 15y^4$$

(*d*)
$$\frac{\partial f}{\partial x} = 16xy^4z^5 \qquad \frac{\partial f}{\partial y} = 32x^2y^3z^5 \qquad \frac{\partial f}{\partial z} = 40x^2y^4z^4$$

(*e*)
$$\frac{\partial z}{\partial w} = 18w - 4x \qquad \frac{\partial z}{\partial x} = -4w + 30x^4 - 3y \qquad \frac{\partial z}{\partial y} = -3x + 14y$$

(*f*)
$$z_w = 39w^2 + 6wx^3y^4 \qquad z_x = 9w^2x^2y^4 - 40x^3 \qquad z_y = 12w^2x^3y^3 - 33y^2$$

9.2 Use the product rule from (*9.2*) to find the first-order partial derivatives for each of the following functions:

(*a*) $f(x, y) = 7x^3(4x + 9y^2)$ (*b*) $z = (2w^5 + 3x^2)(w^3 - 5x^4 + 4y^2)$

(*a*)
$$\frac{\partial f}{\partial x} = 7x^3(4) + (4x + 9y^2)(21x^2) \qquad \frac{\partial f}{\partial y} = 7x^3(18y) + (4x + 9y^2)(0)$$
$$= 112x^3 + 189x^2y^2 \qquad = 126x^3y$$

(*b*)
$$\begin{aligned} z_w &= (2w^5 + 3x^2)(3w^2) + (w^3 - 5x^4 + 4y^2)(10w^4) \\ &= 16w^7 + 9w^2x^2 - 50w^4x^4 + 40w^4y^2 \\ z_x &= (2w^5 + 3x^2)(-20x^3) + (w^3 - 5x^4 + 4y^2)(6x) \\ &= -40w^5x^3 - 90x^5 + 6w^3x + 24xy^2 \\ z_y &= (2w^5 + 3x^2)(8y) + (w^3 - 5x^4 + 4y^2)(0) \\ &= 16w^5y + 24x^2y \end{aligned}$$

9.3 Use the quotient rule from (*9.3*) to find the first-order partial derivatives for the following functions:

(*a*) $z = \dfrac{x^2 + y^2}{5x + 2y}$ (*b*) $z = \dfrac{w^3 + y^2}{8w + 4x + 3y}$

(*a*)
$$\frac{\partial z}{\partial x} = \frac{(5x+2y)(2x) - (x^2+y^2)(5)}{(5x+2y)^2} = \frac{5x^2 + 4xy - 5y^2}{(5x+2y)^2}$$
$$\frac{\partial z}{\partial y} = \frac{(5x+2y)(2y) - (x^2+y^2)(2)}{(5x+2y)^2} = \frac{2y^2 + 10xy - 2x^2}{(5x+2y)^2}$$

(*b*)
$$z_w = \frac{(8w+4x+3y)(3w^2) - (w^3+y^2)(8)}{(8w+4x+3y)^2} = \frac{16w^3 + 12w^2x + 9w^2y - 8y^2}{(8w+4x+3y)^2}$$
$$z_x = \frac{(8w+4x+3y)(0) - (w^3+y^2)(4)}{(8w+4x+3y)^2} = \frac{-4w^3 - 4y^2}{(8w+4x+3y)^2}$$
$$z_y = \frac{(8w+4x+3y)(2y) - (w^3+y^2)(3)}{(8w+4x+3y)^2} = \frac{3y^2 + 16wy + 8xy - 3w^3}{(8w+4x+3y)^2}$$

9.4 Find the first-order partials for the following functions using the generalized power function rule from (*9.4*):

(*a*) $f(x, y) = \sqrt{4x + 7y} = (4x + 7y)^{1/2}$ (*b*) $z = (8x^2 + 3xy^3)^5$

(*a*)
$$f_x = \tfrac{1}{2}(4x+7y)^{-1/2} \cdot 4 \qquad f_y = \tfrac{1}{2}(4x+7y)^{-1/2} \cdot 7$$
$$= 2(4x+7y)^{-1/2} \qquad = 3.5(4x+7y)^{-1/2}$$
$$= \frac{2}{\sqrt{4x+7y}}, \qquad = \frac{3.5}{\sqrt{4x+7y}}$$

(*b*)
$$\frac{\partial z}{\partial x} = 5(8x^2 + 3xy^3)^4 \cdot (16x + 3y^3) \qquad \frac{\partial z}{\partial y} = 5(8x^2 + 3xy^3)^4 \cdot (9xy^2)$$
$$= (80x + 15y^3)(8x^2 + 3xy^3)^4 \qquad = 45xy^2(8x^2 + 3xy^3)^4$$

9.5 Find the first-order partial derivatives for the following natural exponential functions, using (*9.5*):

(*a*) $f(x, y) = e^{3xy}$ (*b*) $z = e^{x^2y^2}$

(*a*)
$$\frac{\partial f}{\partial x} = e^{3xy} \cdot 3y = 3ye^{3xy} \qquad \frac{\partial f}{\partial y} = e^{3xy} \cdot 3x = 3xe^{3xy}$$

(*b*)
$$\frac{\partial z}{\partial x} = e^{x^2y^2} \cdot 2xy^2 = 2xy^2e^{x^2y^2} \qquad \frac{\partial z}{\partial y} = e^{x^2y^2} \cdot 2x^2y = 2x^2ye^{x^2y^2}$$

9.6 Use (*9.6*) to find the first-order partial derivatives for the following natural logarithmic functions:

(*a*) $f(x, y) = \ln|x^2 + y^3|$ (*b*) $z = \ln|5 + 2x^3y^5|$

(*a*)
$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^3} \cdot 2x = \frac{2x}{x^2+y^3} \qquad \frac{\partial f}{\partial y} = \frac{1}{x^2+y^3} \cdot 3y^2 = \frac{3y^2}{x^2+y^3}$$

(*b*)
$$z_x = \frac{1}{5+2x^3y^5} \cdot 6x^2y^5 = \frac{6x^2y^5}{5+2x^3y^5} \qquad z_y = \frac{1}{5+2x^3y^5} \cdot 10x^3y^4 = \frac{10x^3y^4}{5+2x^3y^5}$$

9.7 Use whatever combination of rules is necessary to find the first-order partials for each of the following functions:

(a) $z = 5x^3e^{2xy}$ (b) $z = \dfrac{y}{1+e^x}$ (c) $z = y(1+e^x)^{-1}$

(d) $z = \ln|2x-5y| \cdot e^{4xy}$ (e) $z = \dfrac{7xy}{e^{2x+1}}$ (f) $z = 7xye^{-(2x+1)}$

(a) Using the product rule and natural exponential function rule,

$$z_x = 5x^3 \cdot 2ye^{2xy} + e^{2xy} \cdot 15x^2 = 5x^2e^{2xy}(2xy+3) \qquad z_y = 5x^3 \cdot 2xe^{2xy} + e^{2xy} \cdot 0 = 10x^4e^{2xy}$$

(b) Using the quotient rule and natural exponential function rule,

$$z_x = \frac{(1+e^x)(0) - y(e^x)}{(1+e^x)^2} = \frac{-ye^x}{(1+e^x)^2} \qquad z_y = \frac{(1+e^x)(1) - y(0)}{(1+e^x)^2} = \frac{1}{1+e^x}$$

(c) This is the same function as in (b). Here using the product rule and the generalized power function rule,

$$z_x = y \cdot [-1(1+e^x)^{-2} \cdot e^x] + (1+e^x)^{-1} \cdot 0$$
$$= -ye^x(1+e^x)^{-2} = \frac{-ye^x}{(1+e^x)^2}$$
$$z_y = y \cdot 0 + (1+e^x)^{-1} \cdot 1$$
$$= (1+e^x)^{-1} = \frac{1}{1+e^x}$$

(d) By the product, logarithmic, and exponential function rules,

$$z_x = \ln|2x-5y| \cdot 4ye^{4xy} + e^{4xy}\left(\frac{1}{2x-5y} \cdot 2\right)$$
$$= e^{4xy}\left[4y(\ln|2x-5y|) + \frac{2}{2x-5y}\right]$$
$$z_y = \ln|2x-5y| \cdot 4xe^{4xy} + e^{4xy} \cdot \left(\frac{1}{2x-5y} \cdot -5\right)$$
$$= e^{4xy}\left[4x(\ln|2x-5y|) - \frac{5}{2x-5y}\right]$$

(e) By the quotient and natural exponential function rules,

$$z_x = \frac{e^{2x+1}(7y) - 7xy(2e^{2x+1})}{(e^{2x+1})^2} = \frac{7ye^{2x+1}(1-2x)}{(e^{2x+1})^2} = \frac{7y(1-2x)}{e^{2x+1}} \qquad z_y = \frac{e^{2x+1}(7x) - 7xy(0)}{(e^{2x+1})^2} = \frac{7xe^{2x+1}}{(e^{2x+1})^2} = \frac{7x}{e^{2x+1}}$$

(f) This is the same function as in (e). By the product and natural exponential function rules,

$$z_x = 7xy[-2e^{-(2x+1)}] + e^{-(2x+1)}(7y) = 7ye^{-(2x+1)}(-2x+1) = \frac{7y(1-2x)}{e^{2x+1}} \qquad z_y = 7xy(0) + e^{-(2x+1)}(7x) = 7xe^{-(2x+1)} = \frac{7x}{e^{2x+1}}$$

9.8 The first-order partial derivative measures the rate of change or *slope* of a function with respect to the axis specified by the independent variable. Estimate the slopes of the following functions along the different axes at the points indicated. Practice the use of different notations.

(*a*) $f(x, y) = x^2y^5$, at $(3, 2)$ (*b*) $z = e^{xy}$, at $(1, 2)$ (*c*) $z = 2w^6x^5y^3$, at $(1, 2, 1)$

(*a*)

$$\begin{aligned} f_x(x, y) &= 2xy^5 & f_y(x, y) &= 5x^2y^4 \\ f_x(3, 2) &= 2(3)(2)^5 & f_y(3, 2) &= 5(3)^2(2)^4 \\ &= 192 & &= 720 \end{aligned}$$

(*b*)

$$\frac{\partial z}{\partial x} = e^{xy} \cdot y = ye^{xy} \qquad \frac{\partial z}{\partial y} = e^{xy} \cdot x = xe^{xy}$$

$$\frac{\partial z}{\partial x}(1, 2) = (2)e^{(1)(2)} = 2e^2 \qquad \frac{\partial z}{\partial y}(1, 2) = (1)e^{(1)(2)} = e^2$$

(*c*)

$$\begin{aligned} z_w &= 12w^5x^5y^3 & z_x &= 10w^6x^4y^3 & z_y &= 6w^6x^5y^2 \\ z_w(1, 2, 1) &= 12(32) & z_x(1, 2, 1) &= 10(16) & z_y(1, 2, 1) &= 6(32) \\ &= 384 & &= 160 & &= 192 \end{aligned}$$

SECOND-ORDER PARTIAL DERIVATIVES

9.9 Find the second-order direct partial derivatives for each of the following functions. Practice the use of different notations.

(*a*) $f(x, y) = 6x^3y^5$ (*b*) $f(x, y) = 4x^6 - 3x^2y^2 + 5y^4$ (*c*) $z = (2x + 5y)(7x - 3y)$

(*d*) $z = e^{4x-7y}$ (*e*) $f(x, y, z) = 10x^3y^2z^4$

(*a*)

$$\begin{aligned} f_x(x, y) &= 18x^2y^5 & f_y(x, y) &= 30x^3y^4 \\ f_{xx}(x, y) &= 36xy^5 & f_{yy}(x, y) &= 120x^3y^3 \end{aligned}$$

(*b*)

$$\frac{\partial f}{\partial x} = 24x^5 - 6xy^2 \qquad \frac{\partial f}{\partial y} = -6x^2y + 20y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 120x^4 - 6y^2 \qquad \frac{\partial^2 f}{\partial y^2} = -6x^2 + 60y^2$$

(*c*)

$$\begin{aligned} \frac{\partial z}{\partial x} &= (2x + 5y)(7) + (7x - 3y)(2) & \frac{\partial z}{\partial y} &= (2x + 5y)(-3) + (7x - 3y)(5) \\ &= 28x + 29y & &= 29x - 30y \\ \frac{\partial^2 z}{\partial x^2} &= 28 & \frac{\partial^2 z}{\partial y^2} &= -30 \end{aligned}$$

(*d*)

$$\begin{aligned} z_x &= e^{4x-7y} \cdot 4 = 4e^{4x-7y} & z_y &= e^{4x-7y} \cdot (-7) = -7e^{4x-7y} \\ z_{xx} &= 4e^{4x-7y} \cdot 4 = 16e^{4x-7y} & z_{yy} &= -7e^{4x-7y} \cdot (-7) = 49e^{4x-7y} \end{aligned}$$

(*e*)

$$\begin{aligned} f_x &= 30x^2y^2z^4 & f_y &= 20x^3yz^4 & f_z &= 40x^3y^2z^3 \\ f_{xx} &= 60xy^2z^4 & f_{yy} &= 20x^3z^4 & f_{zz} &= 120x^3y^2z^2 \end{aligned}$$

9.10 The second-order direct partial derivative measures the rate of change or *slope* of the first-order partial derivative with respect to the axis specified by the independent variable. For the following functions, estimate the slopes of the first-order partial derivatives with respect to the same principal axis at the points indicated. Continue to use the different notations.

(*a*) $f(x, y) = 5x^4 - 6x^2y^3 - 4y^3$, at $(3, 2)$ (*b*) $z = e^{3x-4y}$, at $(2, 1)$

(*c*) $f(x, y) = \ln|7x - 4y|$, at $(2, 1)$ (*d*) $z = w^3x^4y^3$, at $(2, 1, 2)$

(*a*)

$$\begin{aligned} f_x(x, y) &= 20x^3 - 12xy^3 \\ f_{xx}(x, y) &= 60x^2 - 12y^3 \\ f_{xx}(3, 2) &= 60(3)^2 - 12(2)^3 \\ &= 444 \end{aligned} \qquad \begin{aligned} f_y(x, y) &= -18x^2y^2 - 12y^2 \\ f_{yy}(x, y) &= -36x^2y - 24y \\ f_{yy}(3, 2) &= -36(3)^2(2) - 24(2) \\ &= -696 \end{aligned}$$

(*b*)

$$\begin{aligned} z_x &= 3e^{3x-4y} \\ z_{xx} &= 9e^{3x-4y} \\ z_{xx}(2, 1) &= 9e^{3(2)-4(1)} \\ &= 9e^2 \end{aligned} \qquad \begin{aligned} z_y &= -4e^{3x-4y} \\ z_{yy} &= 16e^{3x-4y} \\ z_{yy}(2, 1) &= 16e^{3(2)-4(1)} \\ &= 16e^2 \end{aligned}$$

(*c*)

$$\begin{aligned} f_x &= \frac{1}{7x - 4y} \cdot 7 \\ &= \frac{7}{7x - 4y} \end{aligned} \qquad \begin{aligned} f_y &= \frac{1}{7x - 4y} \cdot (-4) \\ &= \frac{-4}{7x - 4y} \end{aligned}$$

By the quotient rule (or generalized power function rule),

$$\begin{aligned} f_{xx} &= \frac{(7x - 4y)(0) - 7(7)}{(7x - 4y)^2} \\ &= \frac{-49}{(7x - 4y)^2} \\ f_{xx}(2, 1) &= \frac{-49}{[7(2) - 4(1)]^2} \\ &= \frac{-49}{100} = -.49 \end{aligned} \qquad \begin{aligned} f_{yy} &= \frac{(7x - 4y)(0) - (-4)(-4)}{(7x - 4y)^2} \\ &= \frac{-16}{(7x - 4y)^2} \\ f_{yy}(2, 1) &= \frac{-16}{[7(2) - 4(1)]^2} \\ &= \frac{-16}{100} = -.16 \end{aligned}$$

(*d*)

$$\begin{aligned} z_w &= 3w^2x^4y^3 \\ z_{ww} &= 6wx^4y^3 \\ z_{ww}(2, 1, 2) &= 6 \cdot 2 \cdot 1 \cdot 8 \\ &= 96 \end{aligned} \qquad \begin{aligned} z_x &= 4w^3x^3y^3 \\ z_{xx} &= 12w^3x^2y^3 \\ z_{xx}(2, 1, 2) &= 12 \cdot 8 \cdot 1 \cdot 8 \\ &= 768 \end{aligned} \qquad \begin{aligned} z_y &= 3w^3x^4y^2 \\ z_{yy} &= 6w^3x^4y \\ z_{yy}(2, 1, 2) &= 6 \cdot 8 \cdot 1 \cdot 2 \\ &= 96 \end{aligned}$$

9.11 Find the cross partial derivatives for each of the following functions. Note how the order of independent variables x and y is reversed in the different notational forms.

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y}$$

(*a*) $f(x, y) = 5x^3y^2 - 10x^2y^4$ (*b*) $z = e^{x^2y^3}$ (*c*) $z = \ln|x^2 + 5y|$

(*d*) $f(x, y, z) = x^3y^{-4}z^{-5}$

(*a*)

$$\frac{\partial f}{\partial x} = 15x^2y^2 - 20xy^4 \qquad \frac{\partial f}{\partial y} = 10x^3y - 40x^2y^3$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 30x^2y - 80xy^3 \qquad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 30x^2y - 80xy^3$$

(*b*)

$$z_x = e^{x^2y^3} \cdot 2xy^3 = 2xy^3e^{x^2y^3} \qquad z_y = e^{x^2y^3} \cdot 3x^2y^2 = 3x^2y^2e^{x^2y^3}$$

By the product rule,

$$\begin{aligned} z_{xy} &= (z_x)_y = 2xy^3 \cdot 3x^2y^2e^{x^2y^3} + e^{x^2y^3} \cdot 6xy^2 = (6x^3y^5 + 6xy^2)e^{x^2y^3} \\ z_{yx} &= (z_y)_x = 3x^2y^2 \cdot 2xy^3e^{x^2y^3} + e^{x^2y^3} \cdot 6xy^2 = (6x^3y^5 + 6xy^2)e^{x^2y^3} \end{aligned}$$

(c)
$$z_x = \frac{2x}{x^2+5y} = 2x(x^2+5y)^{-1} \qquad z_y = \frac{5}{x^2+5y} = 5(x^2+5y)^{-1}$$

By the generalized power function rule (or the quotient rule),

$$\begin{aligned} z_{xy} &= -2x(x^2+5y)^{-2}(5) & \qquad z_{yx} &= -5(x^2+5y)^{-2}(2x) \\ &= \frac{-10x}{(x^2+5y)^2} & &= \frac{-10x}{(x^2+5y)^2} \end{aligned}$$

(d)
$$\begin{aligned} f_x &= 3x^2y^{-4}z^{-5} & \qquad f_y &= -4x^3y^{-5}z^{-5} & \qquad f_z &= -5x^3y^{-4}z^{-6} \\ f_{xy} &= -12x^2y^{-5}z^{-5} & f_{yx} &= -12x^2y^{-5}z^{-5} & f_{zx} &= -15x^2y^{-4}z^{-6} \\ f_{xz} &= -15x^2y^{-4}z^{-6} & f_{yz} &= 20x^3y^{-5}z^{-6} & f_{zy} &= 20x^3y^{-5}z^{-6} \end{aligned}$$

Note how by Young's theorem, $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$.

9.12 The cross partial derivative measures the rate of change or *slope* of the first partial derivative with respect to the axis designated by the other independent variable. For the following functions, estimate the slope of the first partial derivative with respect to the other axis at the points given.

(a) $f(x, y) = 4x^4 - 12x^2y^2 - 5y^3$, at $(-1, 5)$ (b) $f(x, y) = (2x - 5y)^4$, at $(6, 2)$

(c) $z = e^{x^2-8y}$, at $(4, 2)$

(a)
$$\begin{aligned} f_x &= 16x^3 - 24xy^2 & \qquad f_y &= -24x^2y - 15y^2 \\ f_{xy} &= -48xy & f_{yx} &= -48xy \\ f_{xy}(-1, 5) &= 240 & f_{yx}(-1, 5) &= 240 \end{aligned}$$

(b) Using the generalized power function rule twice,

$$\begin{aligned} f_x &= 4(2x-5y)^3(2) & \qquad f_y &= 4(2x-5y)^3(-5) \\ &= 8(2x-5y)^3 & &= -20(2x-5y)^3 \\ f_{xy} &= 24(2x-5y)^2(-5) & f_{yx} &= -60(2x-5y)^2(2) \\ &= -120(2x-5y)^2 & &= -120(2x-5y)^2 \\ f_{xy}(6, 2) &= -480 & f_{yx}(6, 2) &= -480 \end{aligned}$$

(c)
$$\begin{aligned} z_x &= 2xe^{x^2-8y} & \qquad z_y &= -8e^{x^2-8y} \\ z_{xy} &= -16xe^{x^2-8y} & z_{yx} &= -16xe^{x^2-8y} \\ z_{xy}(4, 2) &= -16(4)e^{(0)} & z_{yx}(4, 2) &= -16(4)e^{(0)} \\ &= -64 & &= -64 \end{aligned}$$

OPTIMIZATION OF MULTIVARIABLE FUNCTIONS

9.13 For each of the following functions, (1) find the critical points where the function is optimized and (2) determine whether at these points the function is maximized or minimized.

(a) $f(x, y) = 5x^2 + 4xy + 3y^2 - 52x - 56y + 13$ (b) $f(x, y) = 2x^3 - 4y^2 - 216x + 24y + 7$

(c) $f(x, y) = 4x^3 + 6y^2 - 48xy + 9$ (d) $f(x, y) = 2x^3 - 6x^2 + y^3 + 3y^2 - 48x - 45y$

(a) (1) Take the first-order partial derivatives, set them equal to zero, and solve simultaneously using the methods of Section 2.5.

$$f_x = 10x + 4y - 52 = 0 \tag{9.19}$$

$$f_y = 4x + 6y - 56 = 0 \tag{9.20}$$

$x = 2 \qquad y = 8 \qquad$ Critical point

(2) Take the second-order direct partial derivatives from (*9.19*) and (*9.20*), evaluate them at the critical point, and check their signs.

$$f_{xx} = 10 \qquad f_{yy} = 6$$
$$f_{xx}(2,8) = 10 > 0 \qquad f_{yy}(2,8) = 6 > 0$$

With both second-order direct partial derivatives positive, the function is possibly at a relative minimum. Now take the cross partial from (*9.19*) or (*9.20*),

$$f_{xy} = 4 = f_{yx}$$

evaluate it at the critical point and test the third condition:

$$f_{xy}(2,8) = 4 = f_{yx}(2,8)$$
$$f_{xx}(2,8) \cdot f_{yy}(2,8) > [f_{xy}(2,8)]^2$$
$$10 \quad \cdot \quad 6 \quad > \quad 4^2$$

With $f_{xx}, f_{yy} > 0$ and $f_{xx} f_{yy} > (f_{xy})^2$, the function is at a relative minimum at (2, 8).

(*b*) (1) Take the first-order partials and set them equal to zero.

$$f_x = 6x^2 - 216 = 0 \tag{9.21}$$
$$f_y = -8y + 24 = 0 \tag{9.22}$$

Solve for the critical points.

$$6x^2 = 216 \qquad -8y = -24$$
$$x^2 = 36 \qquad y = 3$$
$$x = \pm 6$$
$$(6,3), (-6,3) \qquad \text{Critical points}$$

(2) From (*9.21*) and (*9.22*), take the second direct partials,

$$f_{xx} = 12x \qquad f_{yy} = -8$$

evaluate them at the critical points and note the signs.

$$f_{xx}(6,3) = 12(6) = 72 > 0 \qquad f_{yy}(6,3) = -8 < 0$$
$$f_{xx}(-6,3) = 12(-6) = -72 < 0 \qquad f_{yy}(6,3) = -8 < 0$$

Then take the cross partial from (*9.21*) or (*9.22*),

$$f_{xy} = 0 = f_{yx}$$

evaluate it at the critical points, and test the third condition.

$$f_{xx}(a,b) \cdot f_{yy}(a,b) > [f_{xy}(a,b)]^2$$

At (6, 3), $\quad 72 \quad \cdot \quad -8 \quad < \quad 0$

At (−6, 3), $\quad -72 \quad \cdot \quad -8 \quad > \quad 0$

With $f_{xx}, f_{yy} < 0$ and $f_{xx} f_{yy} > (f_{xy})^2$ at $(-6,3)$, $f(-6,3)$ is a relative maximum. With f_{xx} and f_{yy} of different signs at $(6,3)$, $f(6,3)$ is a saddle point.

(*c*) (1) Set the first-order partial derivatives equal to zero,

$$f_x = 12x^2 - 48y = 0 \tag{9.23}$$
$$f_y = 12y - 48x = 0 \tag{9.24}$$

and solve for the critical values:

$$\begin{array}{cc} 48y = 12x^2 & 12y = 48x \\ y = \frac{1}{4}x^2 & y = 4x \end{array} \tag{9.25}$$

Setting y equal to y,

$$\begin{gathered}\tfrac{1}{4}x^2 = 4x \\ x^2 - 16x = 0 \\ x(x-16) = 0 \\ x = 0 \qquad x = 16\end{gathered}$$

Substituting $x = 0$ and $x = 16$ in $y = 4x$ from (*9.25*),

$$\begin{gathered} y = 4(0) = 0 \\ y = 4(16) = 64 \end{gathered}$$

Hence $\qquad (0, 0) \qquad (16, 64) \qquad$ Critical points

(2) Take the second direct partials from (*9.23*) and (*9.24*),

$$f_{xx} = 24x \qquad f_{yy} = 12$$

evaluate them at the critical points, and note the signs.

$$\begin{array}{rr} f_{xx}(0,0) = 24(0) = 0 & f_{yy}(0,0) = 12 > 0 \\ f_{xx}(16,64) = 24(16) = 384 > 0 & f_{yy}(16,64) = 12 > 0 \end{array}$$

Then take the cross partial from (*9.23*) or (*9.24*),

$$f_{xy} = -48 = f_{yx}$$

evaluate it at the critical points, and test the third condition.

$$f_{xx}(a,b) \cdot f_{yy}(a,b) > [f_{xy}(a,b)]^2$$

$$\begin{array}{lcccccc} \text{At } (0,0), & 0 & \cdot & 12 & < & (-48)^2 \\ \text{At } (16,64), & 384 & \cdot & 12 & > & (-48)^2 \\ & & & 4608 > 2304 & & \end{array}$$

The function is at a relative minimum at (16, 64) and at neither a relative maximum nor minimum at (0, 0). With $f_{xx} f_{yy} < (f_{xy})^2$ and f_{xx} and f_{yy} of the same sign at (0, 0), the function is at an inflection point at (0, 0). Recall that if $f_{xx} \cdot f_{yy} = (f_{xy})^2$, the test is inconclusive.

(*d*) (1)

$$\begin{array}{rr} f_x = 6x^2 - 12x - 48 = 0 & f_y = 3y^2 + 6y - 45 = 0 \\ 6(x^2 - 2x - 8) = 0 & 3(y^2 + 2y - 15) = 0 \\ (x+2)(x-4) = 0 & (y-3)(y+5) = 0 \\ x = -2 \qquad x = 4 & y = 3 \qquad y = -5 \end{array} \qquad (9.26)$$

Hence $\qquad (-2, 3) \qquad (-2, -5) \qquad (4, 3) \qquad (4, -5) \qquad$ Critical points

(2) Take the second direct partials from (*9.26*), evaluate them at the critical points, and note the signs.

$$\begin{array}{lrr} & f_{xx} = 12x - 12 & f_{yy} = 6y + 6 \\ \text{(i)} & f_{xx}(-2,3) = -36 < 0 & f_{yy}(-2,3) = 24 > 0 \\ \text{(ii)} & f_{xx}(-2,-5) = -36 < 0 & f_{yy}(-2,-5) = -24 < 0 \\ \text{(iii)} & f_{xx}(4,3) = 36 > 0 & f_{yy}(4,3) = 24 > 0 \\ \text{(iv)} & f_{xx}(4,-5) = 36 > 0 & f_{yy}(4,-5) = -24 < 0 \end{array}$$

With different signs in (i) and (iv), (−2, 3) and (4, −5) can be ignored, if desired, as saddle points. Now take the cross partial from (*9.26*) and test the third condition.

$$\begin{gathered} f_{xy} = 0 = f_{yx} \\ f_{xx}(a,b) \cdot f_{yy}(a,b) > [f_{xy}(a,b)]^2 \end{gathered}$$

$$\begin{array}{lccccc} \text{From (ii)}, & (-36) & \cdot & (-24) & > & (0)^2 \\ \text{From (iii)}, & (36) & \cdot & (24) & > & (0)^2 \end{array}$$

The function is at a relative maximum at (−2, −5), at a relative minimum at (4, 3), and at a saddle point at (−2, 3) and (4, −5).

CONSTRAINED OPTIMIZATION AND LAGRANGE MULTIPLIERS

9.14 Use Lagrange multipliers to optimize the following functions subject to the given constraint:

(a) Minimize $f(x, y) = 3x^2 - 4xy + 5y^2$, subject to $x + y = 60$

(b) Maximize $f(x, y) = 20x - 4x^2 + 9xy - 6y^2 + 18y$, subject to $x + 3y = 64$

(c) Maximize $f(x, y, z) = 3x^2yz$, subject to $x + y + z = 32$

(d) Minimize $f(x, y, z) = 4xy + 7xz + 9yz$, subject to $xyz = 2016$

(a) (1) Set the constraint equal to zero, multiply it by λ, and add it to the objective function to obtain

$$F = 3x^2 - 4xy + 5y^2 + \lambda(x + y - 60)$$

(2) Take all the first-order partial derivatives, set them equal to zero, and solve for x, y (and λ if desired).

$$F_x = 6x - 4y + \lambda = 0 \tag{9.27}$$

$$F_y = -4x + 10y + \lambda = 0 \tag{9.28}$$

$$F_\lambda = x + y - 60 = 0 \tag{9.29}$$

Subtracting (*9.28*) from (*9.27*) to eliminate λ gives

$$10x - 14y = 0 \qquad x = 1.4y$$

Substituting $x = 1.4y$ in (*9.29*),

$$1.4y + y = 60 \qquad y_0 = 25$$

Then by substituting $y_0 = 25$ in the previous equations, we find that at the critical point:

$$x_0 = 35 \qquad y_0 = 25 \qquad \lambda_0 = -110$$

(b) Following the same procedure as in (a) throughout,

(1)
$$F = 20x - 4x^2 + 9xy - 6y^2 + 18y + \lambda(x + 3y - 64)$$

(2)
$$F_x = -8x + 9y + 20 + \lambda = 0 \tag{9.30}$$

$$F_y = 9x - 12y + 18 + 3\lambda = 0 \tag{9.31}$$

$$F_\lambda = x + 3y - 64 = 0 \tag{9.32}$$

Multiplying (*9.31*) by $\frac{1}{3}$ and subtracting from (*9.30*) to eliminate λ gives

$$-11x + 13y + 14 = 0 \tag{9.33}$$

Multiplying (*9.32*) by 11 and adding to (*9.33*) to eliminate x,

$$46y - 690 = 0 \qquad y_0 = 15$$

Then substituting $y_0 = 15$ into the previous equations shows that at the critical point:

$$x_0 = 19 \qquad y_0 = 15 \qquad \lambda_0 = -3$$

(c) (1)
$$F = 3x^2yz + \lambda(x + y + z - 32)$$

(2)
$$F_x = 6xyz + \lambda = 0 \tag{9.34}$$

$$F_y = 3x^2z + \lambda = 0 \tag{9.35}$$

$$F_z = 3x^2y + \lambda = 0 \tag{9.36}$$

$$F_\lambda = x + y + z - 32 = 0 \tag{9.37}$$

Equating λ's from (*9.35*) and (*9.36*),

$$3x^2z = 3x^2y \qquad z = y$$

Equating λ's from (*9.35*) and (*9.34*),

$$3x^2z = 6xyz \qquad x = 2y$$

Substituting $z = y$ and $x = 2y$ in (*9.37*),

$$2y + y + y - 32 = 0 \qquad 4y = 32 \qquad y_0 = 8$$

Then substituting $y_0 = 8$ in the previous equations gives

$$x_0 = 16 \qquad y_0 = 8 \qquad z_0 = 8 \qquad \lambda_0 = -6144$$

(*d*) (1)

$$F = 4xy + 7xz + 9yz + \lambda(xyz - 2016)$$

(2)

$$F_x = 4y + 7z + \lambda yz = 0 \tag{9.38}$$

$$F_y = 4x + 9z + \lambda xz = 0 \tag{9.39}$$

$$F_z = 7x + 9y + \lambda xy = 0 \tag{9.40}$$

$$F_\lambda = xyz - 2016 = 0 \tag{9.41}$$

Solving (*9.38*), (*9.39*), and (*9.40*) for $-\lambda$,

$$-\lambda = \frac{4y + 7z}{yz} = \frac{4}{z} + \frac{7}{y} \tag{9.42}$$

$$-\lambda = \frac{4x + 9z}{xz} = \frac{4}{z} + \frac{9}{x} \tag{9.43}$$

$$-\lambda = \frac{7x + 9y}{xy} = \frac{7}{y} + \frac{9}{x} \tag{9.44}$$

Now by equating $-\lambda$'s in (*9.42*) and (*9.43*) we eliminate $4/z$ to get

$$\frac{7}{y} = \frac{9}{x} \qquad 7x = 9y \qquad x = \frac{9}{7}\,y$$

and by equating $-\lambda$'s in (*9.44*) and (*9.43*) we eliminate $9/x$ to get

$$\frac{7}{y} = \frac{4}{z} \qquad 7z = 4y \qquad z = \frac{4}{7}\,y$$

Finally by substituting $x = \frac{9}{7}y$ and $z = \frac{4}{7}y$ in (*9.41*), we obtain

$$\tfrac{9}{7}y \cdot y \cdot \tfrac{4}{7}y = 2016$$
$$y^3 = \tfrac{49}{36} \cdot 2016 = 2744$$
$$y_0 = 14$$

and the critical values are $x_0 = 18$, $y_0 = 14$, $z_0 = 8$, $\lambda_0 = -1$. See also Problems 9.20–9.27.

TOTAL DIFFERENTIALS

9.15 Find the total differentials for the following functions:

(*a*) $z = 7x^2 + 4xy + 12y^2$ (*b*) $z = 15x^4y^5$ (*c*) $z = 8(2x - 5y)^3$

(*d*) $f(x, y, z) = 5x^6y^3z^4$

(*a*) Recalling from (*9.13*) that $dz = z_x \cdot dx + z_y \cdot dy$, simply take the first-order partial derivatives and substitute.

$$z_x = 14x + 4y \qquad z_y = 4x + 24y$$

and

$$dz = (14x + 4y)\,dx + (4x + 24y)\,dy$$

(*b*)

$$z_x = 60x^3y^5 \qquad z_y = 75x^4y^4$$
$$dz = (60x^3y^5)\,dx + (75x^4y^4)\,dy$$

(*c*)

$$z_x = 48(2x - 5y)^2 \qquad z_y = -120(2x - 5y)^2$$
$$dz = 48(2x - 5y)^2\,dx - 120(2x - 5y)^2\,dy$$

(*d*)

$$f_x = 30x^5y^3z^4 \qquad f_y = 15x^6y^2z^4 \qquad f_z = 20x^6y^3z^3$$
$$df = (30x^5y^3z^4)\,dx + (15x^6y^2z^4)\,dy + (20x^6y^3z^3)\,dz$$

PRACTICAL APPLICATIONS

9.16 For each of the following functions (F), (1) estimate the rates of change or marginal functions (F_x, F_y) at the given level (a, b) and (2) explain their significance.

(*a*) Total revenue, $R = 9x^2 - 5xy + 12y^2$, at $(4, 6)$

(*b*) Total cost, $C = 80x^2 + 15xy + 50y^2 + 145$, at $(10, 8)$

(*c*) Cobb-Douglas production function, $Q = K^{1/3}L^{2/3}$, at $(8, 27)$

(*a*) (1) Marginal revenues:

$$R_x = 18x - 5y \qquad R_y = 24y - 5x$$
$$R_x(4, 6) = 42 \qquad R_y(4, 6) = 124$$

(2) If y is held constant at 6, the sale of an extra unit of x adds approximately \$42 to total revenue; if x is held constant at 4, the sale of an extra unit of y adds approximately \$124.

(*b*) (1) Marginal costs:

$$C_x = 160x + 15y \qquad C_y = 100y + 15x$$
$$C_x(10, 8) = 1720 \qquad C_y(10, 8) = 950$$

(2) If y is held constant at 8, the production of an extra unit of x adds approximately \$1720 to total cost; if x is held constant at 10, the production of an extra unit of y adds approximately \$950 to total cost.

(*c*) (1) Marginal products:

$$\text{MP}_K = Q_K = \tfrac{1}{3}K^{-2/3}L^{2/3} \qquad \text{MP}_L = Q_L = \tfrac{2}{3}K^{1/3}L^{-1/3}$$
$$Q_K(8, 27) = \frac{(27)^{2/3}}{3(8)^{2/3}} \qquad Q_L(8, 27) = \frac{2(8)^{1/3}}{3(27)^{1/3}}$$
$$= \tfrac{3}{4} \qquad = \tfrac{4}{9}$$

(2) If L is held constant at 27 units and K is increased by 1 unit, output will go up by approximately $\frac{3}{4}$ unit; if K is held constant at 8 units and L is increased by 1 unit, production will go up by $\frac{4}{9}$ unit.

9.17 (1) Find the critical points at which the following profit functions are maximized and (2) test the second-order conditions.

(*a*) $\pi = 190x - 3x^2 - 5xy - 4y^2 + 235y - 54$

(*b*) $\pi = 266x - 5x^2 - 2xy - 6y^2 + 146y - 104$

(*c*) $\pi = 550x - 8x^2 - 7xy - 12y^2 + 450y - 223$

(*a*) (1) Take the first-order partials, set them equal to zero, and solve simultaneously using the method of Section 2.5.

$$\pi_x = -6x - 5y + 190 = 0$$
$$\pi_y = -5x - 8y + 235 = 0$$
$$x = 15 \qquad y = 20 \qquad \text{Critical point}$$

(2) Take the necessary second partial derivatives,

$$\pi_{xx} = -6 \qquad \pi_{yy} = -8 \qquad \pi_{xy} = -5$$

evaluate them at the critical points (a constant function will remain constant), and test the third condition:

$$\pi_{xx} \cdot \pi_{yy} > (\pi_{xy})^2$$
$$(-6) \cdot (-8) > (-5)^2$$
$$48 > 25$$

With $\pi_{xx}, \pi_{yy} < 0$ and $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$, profits are maximized at $(15, 20)$.

(*b*) (1)

$$\pi_x = -10x - 2y + 266 = 0$$
$$\pi_y = -2x - 12y + 146 = 0$$

Solved simultaneously,

$$x = 25 \qquad y = 8 \qquad \text{Critical point}$$

(2) Taking the second-order partial derivatives,

$$\pi_{xx} = -10 \qquad \pi_{yy} = -12 \qquad \pi_{xy} = -2$$

With π_{xx}, $\pi_{yy} < 0$ and $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$, profits are maximized at (25, 8).

(*c*) (1)

$$\pi_x = -16x - 7y + 550 = 0$$
$$\pi_y = -7x - 24y + 450 = 0$$
$$x = 30 \qquad y = 10 \qquad \text{Critical point}$$

(2)

$$\pi_{xx} = -16 \qquad \pi_{yy} = -24 \qquad \pi_{xy} = -7$$

With π_{xx}, $\pi_{yy} < 0$ and $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$, profits are maximized at (30, 10).

9.18 Given the following demand and joint-cost functions for a monopolist, find (1) the profit-maximizing level of output and (2) the profit-maximizing price.

(*a*) $p_x = 144 - 5x$, $p_y = 148 - 3y$, $\quad C = x^2 + 4xy + y^2 + 75$

(*b*) $p_x = 570 - 7x$, $p_y = 710 - 9y$, $\quad C = 2x^2 + 6xy + 5y^2 + 156$

(*a*) (1) Profit (π) = total revenue (R) − total cost (C), where $R = R_x + R_y$ and $R_x = p_x \cdot x$ and $R_y = p_y \cdot y$. By successive substitutions,

$$\begin{aligned}\pi &= p_x \cdot x + p_y \cdot y - C \\ &= (144 - 5x)x + (148 - 3y)y - (x^2 + 4xy + y^2 + 75) \\ &= -6x^2 + 144x - 4xy + 148y - 4y^2 - 75\end{aligned}$$

Then taking the partial derivatives and optimizing as before,

$$\pi_x = -12x - 4y + 144 = 0$$
$$\pi_y = -4x - 8y + 148 = 0$$
$$x_0 = 7 \qquad y_0 = 15 \qquad \text{Critical point}$$

Testing the second-order conditions,

$$\pi_{xx} = -12 \qquad \pi_{yy} = -8 \qquad \pi_{xy} = -4$$

With π_{xx}, $\pi_{yy} < 0$ and $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$, π is maximized at (7, 15).

(2) Evaluating the demand functions at x_0 and y_0,

$$p_x = 144 - 5(7) = 109 \qquad p_y = 148 - 3(15) = 103$$

(*b*) (1)

$$\begin{aligned}\pi &= (570 - 7x)x + (710 - 9y)y - (2x^2 + 6xy + 5y^2 + 156) \\ &= -9x^2 + 570x - 6xy + 710y - 14y^2 - 156\end{aligned}$$

Taking the partial derivatives and solving simultaneously,

$$\pi_x = -18x - 6y + 570 = 0$$
$$\pi_y = -6x - 28y + 710 = 0$$
$$x_0 = 25 \qquad y_0 = 20 \qquad \text{Critical point}$$

Testing the second-order conditions,

$$\pi_{xx} = -18 \qquad \pi_{yy} = -28 \qquad \pi_{xy} = -6$$

With π_{xx}, $\pi_{yy} < 0$ and $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$, π is maximized at (25, 20).

(2) Evaluating the demand functions at x_0 and y_0,

$$p_x = 570 - 7(25) = 395 \qquad p_y = 710 - 9(20) = 530$$

9.19 A monopolist practices *price discrimination* by selling the same product in separate markets at different prices; for example, a utility company charges one rate to factories and another rate to private residences, or an airline charges one fare for businesspeople who cannot stay over a Saturday night and another fare to vacationers who can. Letting $q = q_1 + q_2$, where q = the total number of units sold and q_1, q_2 = the number of units sold in markets 1 and 2, find the profit-maximizing price in each market, given the following demand and cost functions:

(*a*) $p_1 = 626 - 6q_1,\ p_2 = 476 - 5q_2, \qquad C = 425 + q^2$

(*b*) $p_1 = 880 - 4q_1,\ p_2 = 900 - 7q_2, \qquad C = 500 + 3q^2$

(*a*)

$$\begin{aligned}\pi &= p_1 \cdot q_1 + p_2 \cdot q_2 - C \\ &= (626 - 6q_1)q_1 + (476 - 5q_2)q_2 - C \qquad (9.45)\end{aligned}$$

But $C = 425 + q^2$ and $q = q_1 + q_2$. Hence

$$C = 425 + (q_1 + q_2)^2 = 425 + q_1^2 + 2q_1q_2 + q_2^2$$

Substituting in (*9.45*),

$$\begin{aligned}\pi &= 626q_1 - 6q_1^2 + 476q_2 - 5q_2^2 - (425 + q_1^2 + 2q_1q_2 + q_2^2) \\ &= 626q_1 - 7q_1^2 + 476q_2 - 6q_2^2 - 2q_1q_2 - 425\end{aligned}$$

Taking the first partials and solving simultaneously,

$$\frac{\partial \pi}{\partial q_1} = \pi_1 = -14q_1 - 2q_2 + 626 = 0$$

$$\frac{\partial \pi}{\partial q_2} = \pi_2 = -2q_1 - 12q_2 + 476 = 0$$

$$q_1 = 40 \qquad q_2 = 33 \qquad \text{Critical point}$$

Testing the second-order conditions,

$$\pi_{11} = -14 \qquad \pi_{22} = -12 \qquad \pi_{12} = -2$$

With π_{11}, $\pi_{22} < 0$ and $\pi_{11}\pi_{22} > (\pi_{12})^2$, the function is maximized. Then evaluating the demand functions at q_1 and q_2,

$$p_1 = 626 - 6(40) = 386 \qquad p_2 = 476 - 5(33) = 311$$

(*b*)

$$\begin{aligned}\pi &= p_1 \cdot q_1 + p_2 \cdot q_2 - C \\ &= (880 - 4q_1)q_1 + (900 - 7q_2)q_2 - [500 + 3(q_1^2 + 2q_1q_2 + q_2^2)] \\ &= -7q_1^2 + 880q_1 + 900q_2 - 10q_2^2 - 6q_1q_2 - 500\end{aligned}$$

Optimizing the function,

$$\begin{aligned}\pi_1 &= -14q_1 - 6q_2 + 880 = 0 \\ \pi_2 &= -6q_1 - 20q_2 + 900 = 0\end{aligned}$$

$$q_1 = 50 \qquad q_2 = 30 \qquad \text{Critical point}$$

With $\pi_{11} = -14$, $\pi_{22} = -20$, $\pi_{12} = -6$, π is maximized, and

$$p_1 = 880 - 4(50) = 680 \qquad p_2 = 900 - 7(30) = 690$$

9.20 Find the combination of goods x and y that will minimize costs for a producer facing the following cost functions and production quota constraints:

(*a*) $c = 5x^2 - 3xy + 8y^2 + 95$, subject to $x + y = 64$

(*b*) $c = 6x^2 - 2xy + 5y^2 + 143$, subject to $2x + y = 90$

(*a*) (1) Set the constraint equal to zero, multiply it by λ, and add it to the objective function, to form

$$C = 5x^2 - 3xy + 8y^2 + 95 + \lambda(x + y - 64)$$

(2) Take all the first partial derivatives, including C_λ.

$$C_x = 10x - 3y + \lambda = 0 \tag{9.46}$$

$$C_y = -3x + 16y + \lambda = 0 \tag{9.47}$$

$$C_\lambda = x + y - 64 = 0 \tag{9.48}$$

Subtracting (*9.47*) from (*9.46*) gives

$$13x - 19y = 0 \qquad x = \tfrac{19}{13}y$$

Substituting $x = \frac{19}{13}y$ in (*9.48*),

$$\tfrac{19}{13}y + y - 64 = 0 \qquad \tfrac{32}{13}y = 64 \qquad y_0 = 26$$

Then substituting $y_0 = 26$ in (*9.48*), $x_0 = 38$.

(*b*) Following the same format as above,

(1)
$$C = 6x^2 - 2xy + 5y^2 + 143 + \lambda(2x + y - 90)$$

(2)
$$C_x = 12x - 2y + 2\lambda = 0 \tag{9.49}$$

$$C_y = -2x + 10y + \lambda = 0 \tag{9.50}$$

$$C_\lambda = 2x + y - 90 = 0 \tag{9.51}$$

Multiplying (*9.50*) by 2 and subtracting from (*9.49*) gives

$$16x - 22y = 0 \qquad x = \tfrac{11}{8}y$$

Substituting $x = \frac{11}{8}y$ in (*9.51*),

$$2(\tfrac{11}{8}y) + y - 90 = 0 \qquad \tfrac{15}{4}y = 90 \qquad y_0 = 24$$

Then substituting $y_0 = 24$ in (*9.51*), $x_0 = 33$.

9.21 Find the combination of goods x and y that will maximize profits π for a producer facing the following profit functions and output capacity constraints:

(*a*) $\pi = 240x - 5x^2 - 2xy - 3y^2 + 180y$, subject to $x + y = 45$

(*b*) $\pi = 350x - 4x^2 - 3xy - 6y^2 + 960y$, subject to $x + 3y = 130$

(*a*) (1) Set up the Lagrangian function and take the first partials.

$$\Pi = 240x - 5x^2 - 2xy - 3y^2 + 180y + \lambda(x + y - 45)$$

(2)
$$\Pi_x = -10x - 2y + 240 + \lambda = 0 \tag{9.52}$$

$$\Pi_y = -2x - 6y + 180 + \lambda = 0 \tag{9.53}$$

$$\Pi_\lambda = x + y - 45 = 0 \tag{9.54}$$

Subtracting (*9.53*) from (*9.52*) to eliminate λ,

$$-8x + 4y + 60 = 0 \qquad y = 2x - 15$$

Substituting $y = 2x - 15$ in (*9.54*),

$$x + (2x - 15) - 45 = 0 \qquad x_0 = 20$$

Then substituting $x_0 = 20$ in (*9.54*), $y_0 = 25$.

(*b*) (1)
$$\Pi = 350x - 4x^2 - 3xy - 6y^2 + 960y + \lambda(x + 3y - 130)$$

(2)
$$\Pi_x = -8x - 3y + 350 + \lambda = 0 \tag{9.55}$$

$$\Pi_y = -3x - 12y + 960 + 3\lambda = 0 \tag{9.56}$$

$$\Pi_\lambda = x + 3y - 130 = 0 \tag{9.57}$$

Multiplying (*9.56*) by $\frac{1}{3}$ and subtracting from (*9.55*),

$$-7x + y + 30 = 0 \qquad y = 7x - 30$$

Substituting $y = 7x - 30$ in (*9.57*),

$$x + 3(7x - 30) - 130 = 0 \qquad x_0 = 10$$

and substituting $x_0 = 10$ back in (*9.57*), $y_0 = 40$.

9.22 A homeowner wants to enclose an 800 square foot rectangular area in her yard in which her dog can run. Three sides are to be of wiremesh, the other of stone. The wiremesh costs \$8 a running foot; the stone, \$24. What dimensions will minimize the cost?

As seen in Fig. 6-15, the length of stone $L_s = x$, the length of wire $L_w = x + 2y$. The constraint is $A = x \cdot y = 800$ and the objective function is

$$c = 24x + 8(x + 2y) = 32x + 16y$$

Setting up the Lagrangian function,

$$C = 32x + 16y + \lambda(xy - 800)$$

$$C_x = 32 + \lambda y = 0 \tag{9.58}$$

$$C_y = 16 + \lambda x = 0 \tag{9.59}$$

$$C_\lambda = xy - 800 = 0 \tag{9.60}$$

Solving (*9.58*) and (*9.59*) for $-\lambda$,

$$-\lambda = \frac{32}{y}$$

$$-\lambda = \frac{16}{x}$$

Equating $-\lambda$'s,

$$\frac{32}{y} = \frac{16}{x}$$

$$y = 2x$$

Substituting in (*9.60*),

$$x(2x) = 800$$

$$x = 20 \qquad y = 40$$

as was found in Problem 6.19 using another method.

9.23 An airline will not allow carry-on bundles with length plus girth of more than 108 inches. What are the length L and radius r of a permissible cylindrical package with the greatest volume?

The area of the base is πr^2; the circumference (girth) of the cylinder is $2\pi r$. The objective function is $v = L\pi r^2$ and the constraint is $L + 2\pi r = 108$. Using the Lagrangian function,

$$V = L\pi r^2 + \lambda(L + 2\pi r - 108)$$

$$V_L = \pi r^2 + \lambda = 0 \tag{9.61}$$

$$V_r = 2L\pi r + 2\pi\lambda = 0 \tag{9.62}$$

$$V_\lambda = L + 2\pi r - 108 = 0 \tag{9.63}$$

From (*9.61*) and (*9.62*),

$$-\lambda = \pi r^2$$

$$-\lambda = Lr$$

Equating λ's,

$$\pi r^2 = Lr$$
$$L = \pi r$$

Substituting in (*9.63*),

$$\pi r + 2\pi r = 108$$
$$r = \frac{36}{\pi} \qquad L = 36$$

as was found in Problem 6.14 using another method.

9.24 Find the length x, width y, and height z that a rectangular building with a volume of 153 600 cubic feet should have to minimize heat loss given by

$$h = 6xy + 5xz + 10yz$$

With the constraint $v = xyz = 153\,600$,

$$H = 6xy + 5xz + 10yz + \lambda(xyz - 153\,600)$$
$$H_x = 6y + 5z + \lambda yz = 0 \qquad (9.64)$$
$$H_y = 6x + 10z + \lambda xz = 0 \qquad (9.65)$$
$$H_z = 5x + 10y + \lambda xy = 0 \qquad (9.66)$$
$$H_\lambda = xyz - 153\,600 = 0 \qquad (9.67)$$

Solving (*9.64*), (*9.65*), and (*9.66*) for $-\lambda$,

$$-\lambda = \frac{6y + 5z}{yz} = \frac{6}{z} + \frac{5}{y} \qquad (9.68)$$
$$-\lambda = \frac{6x + 10z}{xz} = \frac{6}{z} + \frac{10}{x} \qquad (9.69)$$
$$-\lambda = \frac{5x + 10y}{xy} = \frac{5}{y} + \frac{10}{x} \qquad (9.70)$$

Equating $-\lambda$'s in (*9.68*) and (*9.69*) to eliminate $6/z$.

$$\frac{5}{y} = \frac{10}{x} \qquad 5x = 10y \qquad x = 2y$$

Equating $-\lambda$'s in (*9.69*) and (*9.70*) to eliminate $10/x$,

$$\frac{6}{z} = \frac{5}{y} \qquad 6y = 5z \qquad z = 1.2y$$

Then substituting $x = 2y$ and $z = 1.2y$ in (*9.67*) and solving,

$$2y \cdot y \cdot 1.2y = 153\,600$$
$$2.4y^3 = 153\,600$$
$$y^3 = 64\,000$$
$$y = 40 \qquad x = 80 \qquad z = 48$$

9.25 Maximize the volume $v = xyz$ of a rectangular shed without flooring of length x, width y, and height z for which the cost of the xz wall is \$4 a square foot, the cost of the yz wall is \$7 a square foot, and the cost of the roof (xy) is \$5 a square foot. Limit expenditures to \$10 500.

$$V = xyz + \lambda[2(4xz) + 2(7yz) + 5xy - 10\,500]$$
$$= xyz + \lambda(8xz + 14yz + 5xy - 10\,500)$$

and
$$V_x = yz + 8\lambda z + 5\lambda y = 0 \tag{9.71}$$
$$V_y = xz + 14\lambda z + 5\lambda x = 0 \tag{9.72}$$
$$V_z = xy + 8\lambda x + 14\lambda y = 0 \tag{9.73}$$
$$V_\lambda = 8xz + 14yz + 5xy - 10\,500 = 0 \tag{9.74}$$

Solving (*9.71*), (*9.72*), and (*9.73*) for $-1/\lambda$,

$$yz = -\lambda(8z + 5y) \qquad -\frac{1}{\lambda} = \frac{8z + 5y}{yz} = \frac{8}{y} + \frac{5}{z} \tag{9.75}$$
$$xz = -\lambda(14z + 5x) \qquad -\frac{1}{\lambda} = \frac{14z + 5x}{xz} = \frac{14}{x} + \frac{5}{z} \tag{9.76}$$
$$xy = -\lambda(8x + 14y) \qquad -\frac{1}{\lambda} = \frac{8x + 14y}{xy} = \frac{8}{y} + \frac{14}{x} \tag{9.77}$$

Equating (*9.75*) and (*9.76*) to eliminate $5/z$,

$$\frac{8}{y} = \frac{14}{x} \qquad x = 1.75y$$

Equating (*9.76*) and (*9.77*) to eliminate $14/x$,

$$\frac{5}{z} = \frac{8}{y} \qquad z = .625y$$

Substituting in (*9.74*) and solving,

$$8(1.75y)(.625y) + 14y(.625y) + 5(1.75y)y - 10\,500 = 0$$
$$26.25y^2 = 10\,500$$
$$y = 20$$

Hence $$x = 35 \qquad y = 20 \qquad z = 12.5$$

9.26 Find the amount of K and L that should be employed to maximize output q given the following Cobb-Douglas production functions and constraints set by P_K, P_L, and budget B:

(*a*) $q = K^{.4}L^{.6}$, $P_K = 8$, $P_L = 6$, $B = 300$ (*b*) $q = K^{.75}L^{.25}$, $P_K = 5$, $P_L = 2$, $B = 400$

(*a*) With $P_K = 8$, $P_L = 6$, $B = 300$, the constraint is

$$8K + 6L = 300 \tag{9.78}$$

Setting up the Lagrangian function and taking the partials,

(1) $$\Pi = K^{.4}L^{.6} + \lambda(8K + 6L - 300)$$
(2) $$\Pi_K = .4K^{-.6}L^{.6} + 8\lambda = 0 \tag{9.79}$$
$$\Pi_L = .6K^{.4}L^{-.4} + 6\lambda = 0 \tag{9.80}$$
$$\Pi_\lambda = 8K + 6L - 300 = 0 \tag{9.81}$$

Solving for $-\lambda$ in (*9.79*) and (*9.80*),

$$-\lambda = .05K^{-.6}L^{.6}$$
$$-\lambda = .1K^{.4}L^{-.4}$$

Equating $-\lambda$'s,

$$.05K^{-.6}L^{.6} = .1K^{.4}L^{-.4}$$

To solve for L in terms of K, multiply both sides of the equation by the product of [K raised to the exponent which when added to $(-.6)$ gives 0] $\cdot$ [L raised to the exponent which when added to (.6) gives 1], namely, $K^{.6}L^{.4}$.

$$(K^{.6}L^{.4})(.05K^{-.6}L^{.6}) = (.1K^{.4}L^{-.4})(K^{.6}L^{.4})$$

By adding exponents,

$$.05L = .1K$$
$$L = 2K$$

Substituting in (*9.81*),

$$K_0 = 15 \qquad L_0 = 30$$

(*b*) (1)
$$\Pi = K^{.75}L^{.25} + \lambda(5K + 2L - 400)$$

(2)
$$\Pi_K = .75K^{-.25}L^{.25} + 5\lambda = 0 \tag{9.82}$$
$$\Pi_L = .25K^{.75}L^{-.75} + 2\lambda = 0 \tag{9.83}$$
$$\Pi_\lambda = 5K + 2L - 400 = 0 \tag{9.84}$$

Solving for and equating $-\lambda$'s in (*9.82*) and (*9.83*),

$$-\lambda = .15K^{-.25}L^{.25}$$
$$-\lambda = .125K^{.75}L^{-.75}$$
$$.15K^{-.25}L^{.25} = .125K^{.75}L^{-.75}$$

Multiplying both sides by $(K^{.25}L^{.75})$,

$$.15L = .125K$$

and solving for K,

$$K = 1.2L$$

Substituting in (*9.84*),

$$K_0 = 60 \qquad L_0 = 50$$

9.27 Maximize the following utility functions subject to the given budgetary constraints:

(*a*) $u = x^{.7}y^{.3}$, $p_x = 56$, $p_y = 15$, $B = 400$ (*b*) $u = x^{.2}y^{.8}$, $p_x = 10$, $p_y = 32$, $B = 600$

(*a*) (1)
$$U = x^{.7}y^{.3} + \lambda(56x + 15y - 400)$$

(2)
$$U_x = .7x^{-.3}y^{.3} + 56\lambda = 0 \tag{9.85}$$
$$U_y = .3x^{.7}y^{-.7} + 15\lambda = 0 \tag{9.86}$$
$$U_\lambda = 56x + 15y - 400 = 0 \tag{9.87}$$

Solving for and equating $-\lambda$'s in (*9.85*) and (*9.86*),

$$.0125x^{-.3}y^{.3} = .02x^{.7}y^{-.7}$$

Multiplying both sides by $(x^{.3}y^{.7})$,

$$.0125y = .02x \qquad y = 1.6x$$

Substituting in (*9.87*),

$$x_0 = 5 \qquad y_0 = 8$$

(*b*) (1)
$$U = x^{.2}y^{.8} + \lambda(10x + 32y - 600)$$

(2)
$$U_x = .2x^{-.8}y^{.8} + 10\lambda = 0 \tag{9.88}$$
$$U_y = .8x^{.2}y^{-.2} + 32\lambda = 0 \tag{9.89}$$
$$U_\lambda = 10x + 32y - 600 = 0 \tag{9.90}$$

Solving for and equating $-\lambda$'s in (*9.88*) and (*9.89*),

$$.02x^{-.8}y^{.8} = .025x^{.2}y^{-.2}$$

Multiplying both sides by $(x^{.8}y^{.2})$,

$$.02y = .025x \qquad y = 1.25x$$

Substituting in (*9.90*),

$$x_0 = 12 \qquad y_0 = 15$$