2.4 Linear Equations

A first-order ***linear*** equation is one of the form

$$x' = a(t)x + f(t). \tag{4.1}$$

If $f(t) = 0$, the equation has the form

$$x' = a(t)x, \tag{4.2}$$

and the linear equation is said to be ***homogeneous***. Otherwise it is inhomogeneous.

The functions $a(t)$ and $f(t)$ in (4.1) are called the ***coefficients*** of the equation. We will sometimes consider equations of the more general form

$$b(t)x' = c(t)x + g(t). \tag{4.3}$$

These are still linear equations, and they can be put into the form (4.1) by dividing by $b(t)$—provided $b(t)$ is not zero. The important point about linear equations is that the unknown function x and its derivative x' both appear alone and only to first order. This means that we do not allow x^2, $(x')^3$, xx', e^x, $\cos(x')$, or anything more complicated than just x and x' to appear in the equation. Thus the equations

$$\begin{aligned} x' &= \sin(t)x, \\ y' &= e^{2t}y + \cos t, \quad \text{and} \\ x' &= (3t+2)x + t^2 - 1 \end{aligned}$$

are all linear, while

$$\begin{aligned} x' &= t\sin(x), \\ y' &= yy', \quad \text{and} \\ y' &= 1 - y^2 \end{aligned}$$

are all nonlinear.

Solution of the homogeneous equation

Linear equations can be solved exactly, and we will show how in this section. We start with the homogeneous equation (4.2). You will notice that this is a separable equation. Separating variables and then integrating gives us

$$\frac{dx}{x} = a(t)\,dt \quad \Longrightarrow \quad \ln|x| = \int a(t)\,dt + C.$$

Exponentiating, we get

$$|x| = e^{\int a(t)\,dt + C} = e^C e^{\int a(t)\,dt}.$$

The constant e^C is positive. We will replace it with the constant A and we will allow it to be positive, negative, or zero, so that we can get rid of the absolute value. Hence the general solution is

$$x(t) = Ae^{\int a(t)\,dt}. \tag{4.4}$$

Example 4.5 Solve

$$x' = \sin(t)x.$$

Using the method for separable equations,

$$\begin{aligned} \frac{dx}{x} &= \sin(t)\,dt \\ \ln|x| &= -\cos(t) + C \\ |x(t)| &= e^{-\cos t + C} = e^{C} e^{-\cos t} \end{aligned}$$

or

$$x(t) = Ae^{-\cos t}.$$

Solution of the inhomogeneous equation

We will illustrate the solution method with an example.

Example 4.6 Newton's law of cooling states that the rate at which an object losses or gains heat is proportional to the difference between the temperature of the object (T) and the temperature of the surrounding medium (A). Mathematically, this translates to

$$T' = -k(T - A),$$

where k is the proportionality constant. Solve this equation, assuming that the ambient temperature A is constant.

If we rewrite the equation as

$$T' + kT = kA, \tag{4.7}$$

then the left-hand side reminds us of the formula for the derivative of a product. In fact, if we multiply the left-hand side of equation (4.7) by e^{kt}, it becomes

$$e^{kt}(T' + kT) = e^{kt}T' + ke^{kt}T = [e^{kt}T]', \tag{4.8}$$

the derivative of a product. Using this, equation (4.7) becomes

$$[e^{kt}T]' = kAe^{kt}. \tag{4.9}$$

We can now integrate both sides of this equation to get

$$e^{kt}T(t) = Ae^{kt} + C, \quad \text{or} \quad T(t) = A + Ce^{-kt}. \tag{4.10}$$

This is the general solution to our linear equation.

That worked pretty well. Can we always do this? Let's start with the general linear equation in (4.1) and go through the same steps. First we rewrite it as

$$x' - ax = f, \tag{4.11}$$

in analogy to (4.7). Next, in analogy to (4.8) and (4.9), we want to multiply equation (4.11) by a function $u(t)$, like e^{kt} in the previous example, which will turn the left-hand side into the derivative of a product. Thus we want

$$(ux)' = u(x' - ax). \tag{4.12}$$

We will call such a function an ***integrating factor***.

Assume for the moment that we have found an integrating factor u. Multiplying (4.11) by u, and using (4.12), we get

$$(ux)' = u(x' - ax) = uf.$$

As we did for equation (4.9) in Example 4.6, we can integrate this directly to get

$$u(t)x(t) = \int u(t)f(t)\,dt + C,$$

or

$$x(t) = \frac{1}{u(t)}\int u(t)f(t)\,dt + \frac{C}{u(t)}, \tag{4.13}$$

which is the general solution to (4.1).

Thus, the key to the method is finding an integrating factor, a function u that satisfies equation (4.12). If we expand both sides of (4.12), this becomes

$$ux' + u'x = ux' - aux.$$

Subtracting ux' from each side, this becomes $u'x = -aux$, which will be satisfied if

$$u' = -au. \tag{4.14}$$

This is a linear homogeneous equation, and, as we saw earlier in (4.4), a particular solution is given by

$$u(t) = e^{-\int a(t)\,dt}. \tag{4.15}$$

(Notice that we do not need the constant A that appears in (4.4) because we only need one particular solution. Any solution to (4.14) will do for the present purpose.)

Summary of the method

We have found a general method of solving arbitrary linear equations.

The equation

$$x' = ax + f. \tag{4.16}$$

can be solved using the following four steps.

1. Rewrite the equation as

$$x' - ax = f.$$

2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt},$$

so that the equation becomes

$$(ux)' = u(x' - ax) = uf. \tag{4.17}$$

3. Integrate this equation to obtain

$$u(t)x(t) = \int u(t)f(t)\,dt + C.$$

4. Solve for $x(t)$.

After you have found the integrating factor u in step 2, it is always a good idea to check that equation (4.17) is satisfied.

Let's look at some examples.

Example 4.18 Find the general solution to the equation

$$x' = x + e^{-t}.$$

Let's go about this very carefully. The first thing to do is to bring the term involving x to the left-hand side,

$$x' - x = e^{-t}. \tag{4.19}$$

Next, since $a(t) = 1$, the integrating factor is

$$u(t) = e^{-\int 1\, dt} = e^{-t}.$$

Multiply equation (4.19) by the integrating factor, getting

$$e^{-t}(x' - x) = e^{-2t}. \tag{4.20}$$

Verify that the left-hand side of (4.20) is the derivative of the product $u(t)x(t) = e^{-t}x(t)$, or

$$[e^{-t}x(t)]' = e^{-t}(x' - x) = e^{-2t}. \tag{4.21}$$

We can now integrate both sides of (4.21),

$$\begin{aligned} e^{-t}x(t) &= \int e^{-2t}\, dt \\ &= -\frac{1}{2}e^{-2t} + C. \end{aligned}$$

Finally, we solve for x by multiplying both sides by e^{t}, getting

$$x(t) = -\frac{1}{2}e^{-t} + Ce^{t}. \tag{4.22}$$

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Example 4.23 Find the general solution of

$$x' = x \sin t + 2te^{-\cos t}$$

and the particular solution that satisfies $x(0) = 1$.

This equation is more clearly in the linear form of (4.16) if we rewrite it as $x' = (\sin t)x + 2te^{-\cos t}$. Again we start to find the solution by rewriting the equation as

$$x' - x \sin t = 2te^{-\cos t}.$$

This time $a(t) = \sin t$, so the integrating factor is

$$u(t) = e^{-\int \sin t\, dt} = e^{\cos t}.$$

Multiplying by u, we get

$$[e^{\cos t}x(t)]' = e^{\cos t}(x' - x \sin t) = 2t.$$

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After integrating both sides we have

$$x(t)e^{\cos t} = 2\int t\,dt = t^2 + C.$$

Therefore, the general solution is

$$x(t) = (t^2 + C)e^{-\cos t}. \tag{4.24}$$

The particular solution we want satisfies $x(0) = 1$, so

$$1 = Ce^{-1} \qquad \text{or} \qquad C = e.$$

Thus the solution to the initial value problem is

$$x(t) = (t^2 + e)e^{-\cos t}.$$

Example 4.25 Find the general solution to

$$x' = x\tan t + \sin t,$$

and find the particular solution that satisfies $x(0) = 2$.

Rewrite the equation as

$$x' - x\tan t = \sin t.$$

Then $a(t) = \tan t$, so an integrating factor is

$$u(t) = e^{-\int \tan t\,dt} = e^{\ln(\cos t)} = \cos t.$$

Multiplying by the integrating factor, we get

$$[x\cos t]' = \cos t\left(x' - x\tan t\right) = \cos t\sin t,$$

so

$$x(t)\cos t = \int \cos t\sin t\,dt = -\frac{\cos^2 t}{2} + C.$$

Finally, we divide by $\cos t$ to get

$$x(t) = -\frac{\cos t}{2} + \frac{C}{\cos t}. \tag{4.26}$$

This is the general solution. To find the particular solution with $x(0) = 2$, we substitute this into the formula for the general solution and compute that $C = 5/2$. Thus our particular solution is

$$x(t) = -\frac{\cos t}{2} + \frac{5}{2\cos t}.$$