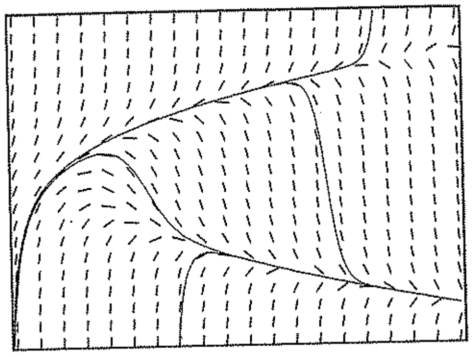
CHAPTER 2



First-Order Equations

In this chapter, we will study first-order equations. We will begin in Section 2.1 by making some definitions and presenting an overview of what we will cover in this chapter. We will then alternate between methods of finding exact solutions and some applications that can be studied using those methods. For each application, we will carefully derive the mathematical models and explore the existence of exact solutions. We will end by showing how qualitative methods can be used to derive useful information about the solutions.

2.1 Differential Equations and Solutions

In this section, we will give an overview of what we want to learn in this chapter. We will visit each topic briefly to give a flavor of what will follow in succeeding sections.

Ordinary differential equations

An ***ordinary differential equation*** is an equation involving an unknown function of a single variable together with one or more of its derivatives. For example, the equation

$$\frac{dy}{dt} = y - t \tag{1.1}$$

is an ordinary differential equation. Here $y = y(t)$ is the unknown function and t is the ***independent variable***.

Some other examples of ordinary differential equations are

$$
\begin{aligned}
y' &= y^2 - t & \qquad ty' &= y \\
y' + 4y &= e^{-3t} & \qquad y' &= \cos(ty) \\
yy'' + t^2 y &= \cos(t) & \qquad y'' &= y^2.
\end{aligned}
\tag{1.2}
$$

The ***order*** of a differential equation is the order of the highest derivative that occurs in the equation. Thus the equation in (1.1) is a ***first-order*** equation since it involves only the first derivative of the unknown function. All of the equations listed in the first two rows of (1.2) are first order. Those in the third row are ***second order*** because they involve the second derivative of y.

The equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \tag{1.3}$$

is not an ordinary differential equation, since the unknown function w is a function of the two independent variables t and x. Because it involves partial derivatives of an unknown function of more than one independent variable, equation (1.3) is called a ***partial differential equation***. For the time being we are interested only in ordinary differential equations.

Normal form

Any first order equation can be put into the form

$$\phi(t, y, y') = 0, \tag{1.4}$$

where ϕ is a function of three variables. For example, the equation in (1.1) can be written as

$$y' - y - t = 0.$$

This equation has the form in (1.4) with $\phi(t, y, z) = z - y - t$. Similarly, the general equation of order n can be written as

$$\phi(t, y, y', \ldots, y^{(n)}) = 0, \tag{1.5}$$

where ϕ is a function of $n + 1$ variables. Notice that all of the equations in (1.2) can be put into this form.

The general forms in (1.4) and (1.5) are too general to deal with in many instances. Frequently we will find it useful to solve for the highest derivative. We will give the result a name.

DEFINITION 1.6

A first-order differential equation of the form

$$y' = f(t, y)$$

is said to be in ***normal form***. Similarly, an equation of order n having the form

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

is said to be in ***normal form***.

Example 1.7 Place the differential equation $t + 4yy' = 0$ into normal form.

This is accomplished by solving the equation $t + 4yy' = 0$ for y'. We find that

$$y' = -\frac{t}{4y}. \tag{1.8}$$

Note that the right-hand side of equation (1.8) is a function of t and y, as required by the normal form $y' = f(t, y)$.

Solutions

A ***solution*** of the first-order, ordinary differential equation $\phi(t, y, y') = 0$ is a differentiable function $y(t)$ such that $\phi(t, y(t), y'(t)) = 0$ for all t in the interval[1] where $y(t)$ is defined.

To discover if a given function is a solution to a differential equation we substitute the function and its derivative(s) into the equation. For example, we can show that $y(t) = t + 1$ is a solution to equation (1.1) by substitution. It is only necessary to compute both sides of equation (1.1) and show that they are equal. We have

$$y'(t) = 1, \quad \text{and} \quad y(t) - t = t + 1 - t = 1.$$

Since the left- and right-hand sides are equal, $y(t) = t + 1$ is a solution.

The process of verifying that a given function is or is not a solution to a differential equation is a very important skill. You can use it to check that your homework solutions are correct. We will use it repeatedly for a variety of purposes, including finding solution methods. Here are two more examples.

Example 1.9 Show that $y(t) = Ce^{-t^2}$ is a solution of the first-order equation

$$y' = -2ty, \tag{1.10}$$

where C is an arbitrary real number.

We compute both sides of the equation and compare them. On the left, we have $y'(t) = -2tCe^{-t^2}$, and on the right, $-2ty(t) = -2tCe^{-t^2}$, so the equation is satisfied. Both $y(t)$ and $y'(t)$ are defined on the interval $(-\infty, \infty)$. Therefore, for each real number C, $y(t) = Ce^{-t^2}$ is a solution of equation (1.10) on the interval $(-\infty, \infty)$.

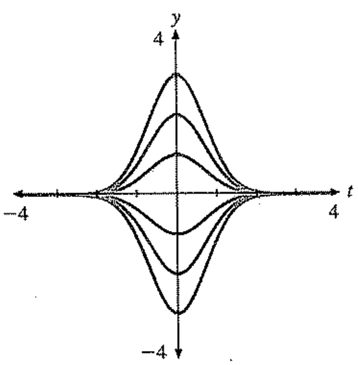


Figure 1. Several solutions to equation (1.10).

Example 1.9 illustrates the fact that a differential equation can have lots of solutions. The solution formula $y(t) = Ce^{-t^2}$ gives a different solution for very value of the constant C. We will see in Section 2.4 that every solution to equation (1.10) is of this form for some value of the constant C. For this reason the formula $y(t) = Ce^{-t^2}$ is called the ***general solution*** to (1.10). The graphs of these solutions are called ***solution curves***, several of which are drawn in Figure 1.

Example 1.11 Is the function $y(t) = \cos t$ a solution to the differential equation $y' = 1 + y^2$?

We substitute $y(t) = \cos t$ into the equation. On the left-hand side we have $y' = -\sin t$. On the right-hand side, $1 + y^2 = 1 + \cos^2 t$. Since $-\sin t \neq 1 + \cos^2 t$ for most values of t, $y(t) = \cos t$ is not a solution.

[1] We will use the notation (a, b), $[a, b]$, $(a, b]$, $[a, b)$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$, and $(-\infty, \infty)$ for intervals. For example, $(a, b) = \{t : a < t < b\}$, $[a, b) = \{t : a \le t < b\}$, $(-\infty, b] = \{t : t \le b\}$, and so on.

Initial value problems

In Example 1.9, we found a general solution, indicated by the presence of an undetermined constant in the formula. This reflects the fact that an ordinary differential equation has infinitely many solutions. In applications, it is necessary to use other information, in addition to the differential equation, to determine the value of the constant and to specify the solution completely. Such a solution is called a ***particular solution***.

Example 1.12 Given that

$$y(t) = -\frac{1}{t - C} \tag{1.13}$$

is a general solution of $y' = y^2$, find the particular solution satisfying $y(0) = 1$.

Because

$$1 = y(0) = \frac{-1}{0 - C} = \frac{1}{C},$$

$C = 1$. Substituting $C = 1$ in equation (1.13) makes

$$y(t) = -\frac{1}{t - 1}, \tag{1.14}$$

a particular solution of $y' = y^2$, satisfying $y(0) = 1$.

DEFINITION 1.15 A first-order differential equation together with an initial condition,

$$y' = f(t, y), \quad y(t_0) = y_0, \tag{1.16}$$

is called an ***initial value problem***. A solution of the initial value problem is a differentiable function $y(t)$ such that

1. $y'(t) = f(t, y(t))$ for all t in an interval containing t_0 where $y(t)$ is defined, and
2. $y(t_0) = y_0$.

Thus, in Example 1.12, the function $y(t) = 1/(1 - t)$ is the solution to the initial value problem

$$y' = y^2, \quad \text{with} \quad y(0) = 1.$$

Interval of existence

The ***interval of existence*** of a solution to a differential equation is defined to be the largest interval over which the solution can be defined and remain a solution. It is important to remember that solutions to differential equations are required to be differentiable, and this implies that they are continuous. The solution to the initial value problem in Example 1.12 is revealing.

Example 1.17 Find the interval of existence for the solution to the initial value problem

$$y' = y^2 \quad \text{with} \quad y(0) = 1.$$

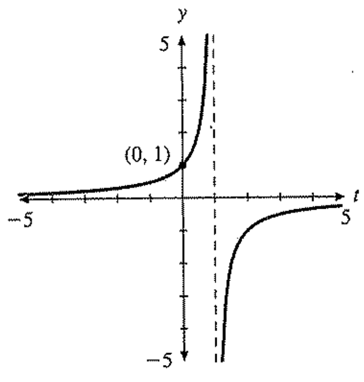


Figure 2. The graph of $y = -1/(t-1)$.

In Example 1.12, we found that the solution is

$$y(t) = \frac{-1}{t-1}.$$

The graph of y is a hyperbola with two branches, as shown in Figure 2. The function y has an infinite discontinuity at $t = 1$. Consequently, this function cannot be considered to be a solution to the differential equation $y' = y^2$ over the whole real line.

Note that the left branch of the hyperbola in Figure 2 passes through the point $(0, 1)$, as required by the initial condition $y(0) = 1$. Hence, the left branch of the hyperbola is the solution curve needed. This particular solution curve extends indefinitely to the left, but rises to positive infinity as it approaches the asymptote $t = 1$ from the left. Any attempt to extend this solution to the right would have to include $t = 1$, at which point the function $y(t)$ is undefined. Consequently, the maximum interval on which this solution curve is defined is the interval $(-\infty, 1)$. This is the interval of existence. ◉

Using variables other than y and t

So far all of our examples have used y as the unknown function, and t as the independent variable. It is not required to use y and t. We can use any letter to designate the independent variable and any other for the unknown function. For example, the equation

$$y' = x + y$$

has the form $y' = f(x, y)$, making x the independent variable and requiring a solution y that is a function of x. This equation has general solution

$$y(x) = -1 - x + Ce^x,$$

which exists on $(-\infty, \infty)$.

Similarly, in the equation

$$s' = \sqrt{r},$$

the independent variable is r and the unknown function is s, so s must be a function of r. The general solution of this equation is

$$s(r) = \frac{2}{3}r^{3/2} + C.$$

This general solution exists on the interval $[0, \infty)$.

Example 1.18 Verify that $x(s) = 2 - Ce^{-s}$ is a solution of

$$x' = 2 - x \tag{1.19}$$

for any constant C. Find the solution that satisfies the initial condition $x(0) = 1$. What is the interval of existence of this solution?

We evaluate both sides of (1.19) for $x(s) = 2 - Ce^{-s}$.

$$x'(s) = Ce^{-s}$$
$$2 - x = 2 - (2 - Ce^{-s}) = Ce^{-s}$$

They are the same, so the differential equation is solved for all $s \in (-\infty, \infty)$. In addition,

$$x(0) = 2 - Ce^{-0} = 2 - C.$$

To satisfy the initial condition $x(0) = 1$, we must have $2 - C = 1$, or $C = 1$. Therefore, $x(s) = 2 - e^{-s}$ is a solution of the initial value problem. This solution exists for all $s \in (-\infty, \infty)$. Its graph is displayed in Figure 3.

Finally, both $x(s)$ and $x'(s)$ exist and solve the equation on $(-\infty, \infty)$. Therefore, the interval of existence is the whole real line. ◈

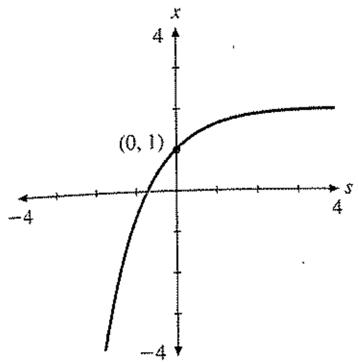


Figure 3. Solution of $x' = 2 - x$, $x(0) = 1$.

The geometric meaning of a differential equation and its solutions

Consider the differential equation

$$y' = f(t, y),$$

where the right-hand side $f(t, y)$ is defined for (t, y) in the rectangle

$$R = \{(t, y) \mid a \le t \le b \text{ and } c \le y \le d\}.$$

Let $y(t)$ be a solution of the equation $y' = f(t, y)$, and recall that the graph of the function y is called a solution curve. Because $y(t_0) = y_0$, the point (t_0, y_0) is on the solution curve. The differential equation says that $y'(t_0) = f(t_0, y_0)$. Hence $f(t_0, y_0)$ is the ***slope*** of any solution curve that passes through the point (t_0, y_0).

This interpretation allows us a new, geometric insight into a differential equation. Consider, if you can, a small, slanted line segment with slope $f(t, y)$ attached to every point (t, y) of the rectangle R. The result is called a ***direction field***, because at each (t, y) there is assigned a direction represented by the line with slope $f(t, y)$.

Even for a simple equation like

$$y' = y, \tag{1.20}$$

it is difficult to visualize the direction field. However, a computer can calculate and plot the direction field at a large number of points—a large enough number for us to get a good understanding of the direction field. Each of the standard mathematics programs, Maple, *Mathematica*, and MATLAB®, has the capability to easily produce direction fields. Some hand-held calculators also have this capability. The student will find that the use of computer- or calculator-generated direction fields will greatly assist their understanding of differential equations. A computer-generated direction field for equation (1.20) is given in Figure 4.

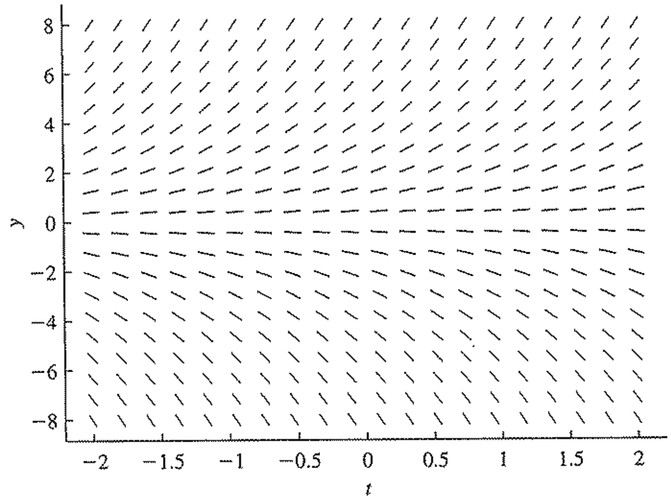


Figure 4. The direction field for $y' = y$.