Chapter 8

Integration

8.1 ANTIDIFFERENTIATION

Chapters 4 through 6 were devoted to differentiation, the process of finding the derivative $F'(x)$ of a function $F(x)$. Frequently, however, we are given the derivative $F'(x)$ and asked to find the original function $F(x)$. Reversing the process of differentiation and finding the original function from the derivative is called *integration* or *antidifferentiation*, and the function $F(x)$ is called the *integral* or *antiderivative* of $F'(x)$.

EXAMPLE 1. Letting $f(x) = F'(x)$ for simplicity, the antiderivative of $f(x)$ is expressed mathematically as

$$\int f(x)\,dx = F(x) + c$$

Here the left-hand side of the equation is read "the *indefinite integral* of f of x with respect to x." The symbol $\int$ is an *integral sign*, $f(x)$ is the *integrand*, and c is the *constant of integration*, which will be explained in Example 3.

8.2 RULES FOR INDEFINITE INTEGRALS

The following rules for indefinite integrals are obtained by reversing the corresponding rules of differentiation. Their accuracy is easily checked by taking the derivative of the antiderivative to be sure it equals the integrand. The rules are illustrated in Example 2 and Problems 8.1–8.3 and 8.9–8.10.

RULE 1. The integral of a constant k is

$$\int k\,dx = kx + c \tag{8.1}$$

RULE 2. The integral of 1, written simply as dx, not $1\,dx$, is

$$\int dx = x + c \tag{8.2}$$

RULE 3. The integral of a power function x^n, where $n \neq -1$, is

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad n \neq -1 \tag{8.3}$$

RULE 4. The integral of x^{-1}(or $1/x$) is

$$\int x^{-1}\,dx = \ln x + c \qquad x > 0 \tag{8.4}$$

The condition $x > 0$ is added because only positive numbers have logarithms. For negative numbers,

$$\int x^{-1}\,dx = \ln|x| + c \qquad x \neq 0 \tag{8.4a}$$

RULE 5. The integral of a natural exponential function is

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c \tag{8.5}$$

RULE 6. The integral of a constant times a function equals the constant times the antiderivative of the function.

$$\int kf(x)\,dx = k\int f(x)\,dx \tag{8.6}$$

RULE 7. The integral of the sum or difference of two or more functions equals the sum or difference of their integrals.

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \tag{8.7}$$

RULE 8. The integral of the negative of a function equals the negative of the integral of the function.

$$\int -f(x)\,dx = -\int f(x)\,dx \tag{8.8}$$

EXAMPLE 2. The rules for indefinite integrals are illustrated below. Check each answer on your own by making sure that the derivative of the antiderivative equals the integrand.

(*a*)
$$\int 5\,dx = 5x + c \qquad \text{[Rule 1]}$$

(*b*)
$$\int x^3\,dx = \frac{1}{3+1}x^{3+1} + c = \frac{1}{4}x^4 + c \qquad \text{[Rule 3]}$$

(*c*)
$$\int 3x^2\,dx = 3\int x^2\,dx \qquad \text{[Rule 6]}$$
$$= 3\left(\frac{1}{2+1}x^{2+1} + c_1\right) \qquad \text{[Rule 3]}$$
$$= x^3 + c$$

where c_1 and c are arbitrary constants and $3c_1 = c$. Since c is an arbitrary constant, it should be ignored in the preliminary calculations and included only in the final solution.

(*d*)
$$\int (1-x)\,dx = \int dx - \int x\,dx \qquad \text{[Rules 2 and 8]}$$
$$= x - \left(\frac{1}{1+1}x^{1+1}\right) \qquad \text{[Rule 3]}$$
$$= x - \tfrac{1}{2}x^2 + c$$

(*e*)
$$\int 15x^{-1}\,dx = 15\int x^{-1}\,dx \qquad \text{[Rule 6]}$$
$$= 15\ln|x| + c \qquad \text{[Rule 4]}$$

(*f*)
$$\int x^{-3}\,dx = \frac{1}{-3+1}x^{-3+1} = -\frac{1}{2}x^{-2} \qquad \text{[Rule 3]}$$

(*g*)
$$\int 25e^{-5x}\,dx = 25\cdot(-\tfrac{1}{5}\cdot e^{-5x}) = -5e^{-5x} \qquad \text{[Rule 5]}$$

EXAMPLE 3. From the rules of differentiation, we know that functions which differ only by a constant k have the same derivative. The function $F(x) = 5x + k$, for instance, has the same derivative, $F'(x) = f(x) = 5$, for any infinite number of possible values for k. If the process is reversed, it is clear that $\int 5\,dx$ must be the antiderivative for an infinite number of functions differing from each other only by a constant. The constant of integration c, then, serves to represent the value of any constant that was part of the original function but precluded from the derivative by the rules of differentiation.

8.3 AREA UNDER A CURVE

There is no geometrical formula for the area under an irregularly shaped curve, such as $y = f(x)$ between $x = a$ and $x = b$ in Fig. 8-1(a). If the interval $[a, b]$ is divided into n subintervals and rectangles erected such that the height of each, for instance, is equal to the largest value of the function in the subinterval, as in Fig. 8-1(b), the sum of the areas of the rectangles $\Sigma_{i=1}^{n} [f(x_i)\,\Delta x_i]$, called a *Riemann sum*, will approximate, but overestimate, the actual area under the curve. The smaller each subinterval (Δx_i), the more rectangles are created and the closer the combined area of the rectangles $\Sigma_{i=1}^{n} [f(x_i)\,\Delta x_i]$ approaches the actual area under the curve. If the number of subintervals is increased so that $n \to \infty$, each subinterval becomes infinitesimal ($\Delta x_i = dx_i = dx$) and the area A under the curve can be expressed mathematically as

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i \tag{8.9}$$

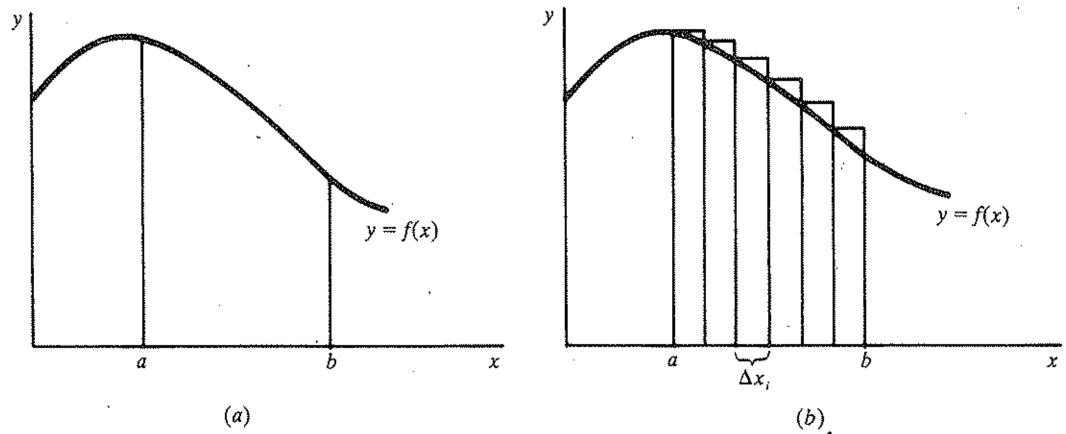


Fig. 8-1

8.4 THE DEFINITE INTEGRAL

The area under a graph of a continuous function such as that in Fig. 8-1 can be expressed more succinctly as the *definite integral* of $f(x)$ over the interval a to b. Put mathematically,

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i \tag{8.10}$$

Here the left-hand side is read "the integral from a to b of f of $x\,dx$." a is called the *lower limit* of integration, b the *upper limit* of integration. Unlike the indefinite integral which is a set of functions containing all the antiderivatives of $f(x)$, as explained in Example 3, the definite integral is a real number which can be evaluated, using the fundamental theorem of calculus.

8.5 THE FUNDAMENTAL THEOREM OF CALCULUS

The *fundamental theorem of calculus* states that the numerical value of the definite integral of a continuous function $f(x)$ over the interval from a to b is given by the antiderivative $F(x) + c$ evaluated at the upper limit of integration b, minus the same antiderivative $F(x) + c$ evaluated at the lower limit of integration a. With c common to both, the constant of integration is eliminated in the subtraction. Expressed mathematically,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \tag{8.11}$$

where the symbol $|_a^b$, $]_a^b$, or $[\cdots]_a^b$ indicates that b and a are to be substituted successively for x. See Examples 4 and 5 and Problem 8.4.

EXAMPLE 4. The definite integrals given below

$$(a)\quad \int_1^5 4x\,dx \qquad (b)\quad \int_2^4 (9x^2+6)\,dx$$

are evaluated as follows:

(a) $$\int_1^5 4x\,dx = 2x^2\Big|_1^5 = 2(5)^2 - 2(1)^2 = 48$$

(b) $$\begin{aligned}\int_2^4 (9x^2+6)\,dx &= [3x^3+6x]_2^4 \\ &= [3(4)^3+6(4)] - [3(2)^3+6(2)] \\ &= 216 - 36 = 180\end{aligned}$$

EXAMPLE 5. The definite integral is used below to determine the area under the curve in Fig. 8-2 over the interval 0 to 5 as follows:

$$A = \int_0^5 (20-4x)\,dx = (20x - 2x^2)\Big|_0^5 = 100 - 50 = 50$$

The answer can easily be checked by using the geometrical formula $A = \frac{1}{2}wh$, where w = width and h = height.

$$A = \tfrac{1}{2}wh = \tfrac{1}{2}xy = \tfrac{1}{2}(5)(20) = 50$$

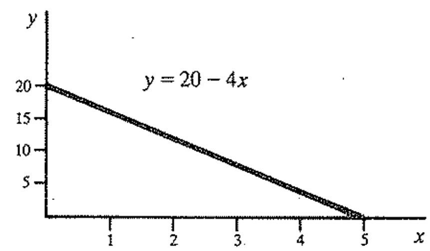


Fig. 8-2

8.6 PROPERTIES OF DEFINITE INTEGRALS AND AREA BETWEEN CURVES

1. Reversing the order of the limits of integration changes the sign of the definite integral.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{8.12}$$

2. If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero.

$$\int_a^a f(x)\,dx = F(a) - F(a) = 0 \tag{8.13}$$

3. The definite integral can be expressed as the sum of component subintegrals.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \qquad a \le b \le c \tag{8.14}$$

4. The sum or difference of two definite integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions.

$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b [f(x) \pm g(x)]\,dx \tag{8.15}$$

5. The definite integral of a constant times a function is equal to the constant times the definite integral of the function.

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \tag{8.16}$$

For the area between curves, see Example 6.

EXAMPLE 6. Using the properties of integrals, the area of the region between two functions such as $y_1 = 3x^2 - 6x + 8$ and $y_2 = -2x^2 + 4x + 1$ from $x = 0$ to $x = 2$, is found in the following way:

(a) Draw a rough sketch of the graph of the functions and shade in the desired area as in Fig. 8-3.

(b) Note the relationship between the curves. Since y_1 lies above y_2, the desired region is simply the area under y_1 minus the area under y_2 between $x = 0$ and $x = 2$. Hence,

$$A = \int_0^2 (3x^2 - 6x + 8)\,dx - \int_0^2 (-2x^2 + 4x + 1)\,dx$$

From (8.15),

$$A = \int_0^2 [(3x^2 - 6x + 8) - (-2x^2 + 4x + 1)]\,dx$$

$$= \int_0^2 (5x^2 - 10x + 7)\,dx$$

$$= \left(\tfrac{5}{3}x^3 - 5x^2 + 7x\right)\Big|_0^2 = [(7\tfrac{1}{3}) - (0)] = 7\tfrac{1}{3}$$

See Problems 8.5–8.6 and 8.30.

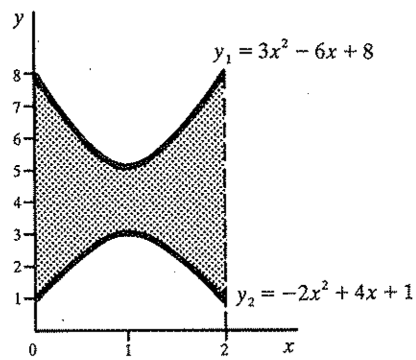


Fig. 8-3

8.7 ESTIMATING DEFINITE INTEGRALS WITH RIEMANN SUMS

Despite the existence of integration tables providing formulas for the integrals of as many as 500 different functions, it is not always easy or possible to find the antiderivative of a function. In such cases, a Riemann sum, $\Sigma_{i=1}^{n} [f(x_i)\,\Delta x_i]$, explained in Section 8.3, can be of help in approximating a definite integral. While any point in the subinterval may be selected for x_i when using a Riemann sum, the midpoints or endpoints are usually chosen, as is illustrated in Examples 7 and 8 and Problems 8.7–8.8.

EXAMPLE 7. If the interval $a \le x \le b$ is to be partitioned into n equal subintervals, where $a = 1$, $b = 4$, and $n = 5$, (*a*) the length of the subintervals (Δx_i), (*b*) the endpoints (g_i), and (*c*) the midpoints (h_i) are determined in the following way:

(*a*) The length of the subintervals (Δx_i) is

$$\Delta x_i = \frac{b-a}{n} = \frac{4-1}{5} = \frac{3}{5} = .6$$

(*b*) The first endpoint is the lower limit: $g_0 = a = 1$. Each subsequent endpoint is Δx units to the right and is found in either of two ways:

$$\begin{aligned} g_1 &= g_0 + \Delta x = 1 + .6 = 1.6 \\ g_2 &= g_0 + 2\,\Delta x = 1 + 2(.6) = 2.2 \\ g_3 &= g_0 + 3\,\Delta x = 1 + 3(.6) = 2.8 \\ g_4 &= g_0 + 4\,\Delta x = 1 + 4(.6) = 3.4 \\ g_5 &= g_0 + 5\,\Delta x = 1 + 5(.6) = 4.0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} g_1 &= g_0 + \Delta x = 1 + .6 = 1.6 \\ g_2 &= g_1 + \Delta x = 1.6 + .6 = 2.2 \\ g_3 &= g_2 + \Delta x = 2.2 + .6 = 2.8 \\ g_4 &= g_3 + \Delta x = 2.8 + .6 = 3.4 \\ g_5 &= g_4 + \Delta x = 3.4 + .6 = 4.0 \end{aligned}$$

(*c*) The first midpoint h_1 is situated half a subinterval length ($\Delta x/2$) from g_0:

$$h_1 = g_0 + \frac{\Delta x}{2} = 1 + \frac{.6}{2} = 1.3$$

Each subsequent midpoint is Δx units to the right:

$$\begin{aligned} h_2 &= h_1 + \Delta x = 1.3 + .6 = 1.9 \\ h_3 &= h_1 + 2\,\Delta x = 1.3 + 2(.6) = 2.5 \\ h_4 &= h_1 + 3\,\Delta x = 1.3 + 3(.6) = 3.1 \\ h_5 &= h_1 + 4\,\Delta x = 1.3 + 4(.6) = 3.7 \end{aligned} \qquad \text{or} \qquad \begin{aligned} h_2 &= h_1 + \Delta x = 1.3 + .6 = 1.9 \\ h_3 &= h_2 + \Delta x = 1.9 + .6 = 2.5 \\ h_4 &= h_3 + \Delta x = 2.5 + .6 = 3.1 \\ h_5 &= h_4 + \Delta x = 3.1 + .6 = 3.7 \end{aligned}$$

See Fig. 8-4.

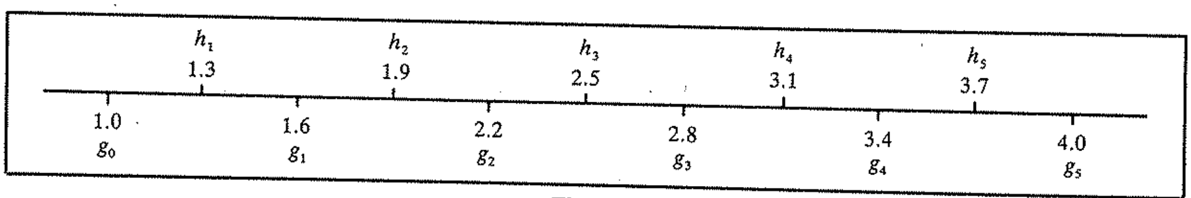


Fig. 8-4

EXAMPLE 8. With $n = 5$ and the midpoint chosen for x_i, a Riemann sum is used to approximate $\int_1^4 x^{-2}\,dx$, as follows:

(*a*) From Example 7, the midpoints were found to be 1.3, 1.9, 2.5, 3.1, and 3.7; the length of the subintervals (Δx_i) was .6.

(*b*) Adapting from (*8.9*),

$$A \approx \sum_{i=1}^{5} [f(x_i)\,\Delta x]$$

$$\approx \left[\frac{1}{(x_1)^2} + \frac{1}{(x_2)^2} + \frac{1}{(x_3)^2} + \frac{1}{(x_4)^2} + \frac{1}{(x_5)^2}\right]\Delta x$$

Substitute the values from (*a*),

$$A \approx \left[\frac{1}{(1.3)^2} + \frac{1}{(1.9)^2} + \frac{1}{(2.5)^2} + \frac{1}{(3.1)^2} + \frac{1}{(3.7)^2}\right](.6)$$

Rounding to three places,

$$A \approx (.592 + .277 + .160 + .104 + .073)(.6) \approx .724$$

To check the accuracy of the approximation, integrate.

$$A = \int_1^4 x^{-2}\,dx = -x^{-1}\Big|_1^4 = -\frac{1}{x}\Big|_1^4 = \left[-\frac{1}{4} - (-1)\right] = .75$$

Increasing the number of subintervals n in a Riemann sum further increases the accuracy of approximation.

8.8 AVERAGE VALUE OF A FUNCTION AND THE VOLUME OF A SOLID OF REVOLUTION

If $f(x)$ is a continuous function on the interval $[a, b]$, the *average value* m of $f(x)$ on this interval is defined as

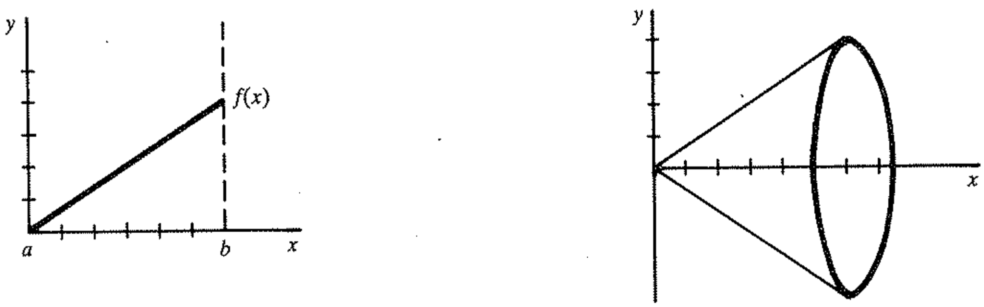
$$m = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{8.17}$$

It is illustrated in Example 9 and Problems 8.11 and 8.31–8.34.

If a continuous function such as $f(x)$ from $x = a$ to $x = b$ in Fig. 8-5(a) is rotated around the x-axis, a solid of revolution is created as is illustrated in Fig. 8-5(b). The *volume of the solid of revolution*, V, can be expressed mathematically as

$$V = \int_a^b \pi[f(x)]^2\,dx \tag{8.18}$$

as is demonstrated in Example 10 and Problems 8.12–8.15.



(a) (b)

Fig. 8-5

EXAMPLE 9. The average value m of $f(x) = x^2 - 1$ from $x = 0$ to $x = 3$ is easily computed using (*8.17*),

$$m = \frac{1}{3-0}\int_0^3 (x^2 - 1)\,dx$$
$$= \tfrac{1}{3}(\tfrac{1}{3}x^3 - x)\Big|_0^3 = \tfrac{1}{3}(9 - 3 - 0) = 2$$

EXAMPLE 10. The volume of the solid of revolution V created by rotating the area bounded by $y = \frac{2}{3}x$ about the x-axis from $x = 0$ to $x = 9$ is readily estimated using (*8.18*),

$$V = \int_0^9 \pi(\tfrac{2}{3}x)^2\,dx$$

Squaring the integrand and rearranging constants,

$$V = \tfrac{4}{9}\pi\int_0^9 x^2\,dx = \tfrac{4}{9}\pi(\tfrac{1}{3}x^3)\Big|_0^9$$
$$= \tfrac{4}{27}\pi(729 - 0) = 108\pi$$

For a graph, consider Fig. 8-5 with $f(x) = \frac{2}{3}x$, $a = 0$, and $b = 9$.

8.9 PRACTICAL APPLICATIONS

Scientists, economists, and business people frequently have information reflecting rates of change, yet sometimes prefer to know the accumulated changes over time. In such circumstances, integration, which reverses the process of differentiation, is helpful. See Examples 11–14 and Problems 8.16–8.34.

EXAMPLE 11. *Drug sensitivity* is measured by the rate of a person's reaction to a particular drug. If the rate of change of temperature T with respect to a dosage x of medicine is given by

$$T'(x) = 3x - .75x^2 \qquad 0 \le x \le 4$$

the full strength of the reaction for the first two units of the medicine is found as follows:

$$\begin{aligned} T(x) &= \int_0^2 (3x - .75x^2)\,dx \\ &= (1.5x^2 - .25x^3)\big|_0^2 \\ &= 6 - 2 = 4 \text{ degrees} \end{aligned}$$

See also Problem 8.18.

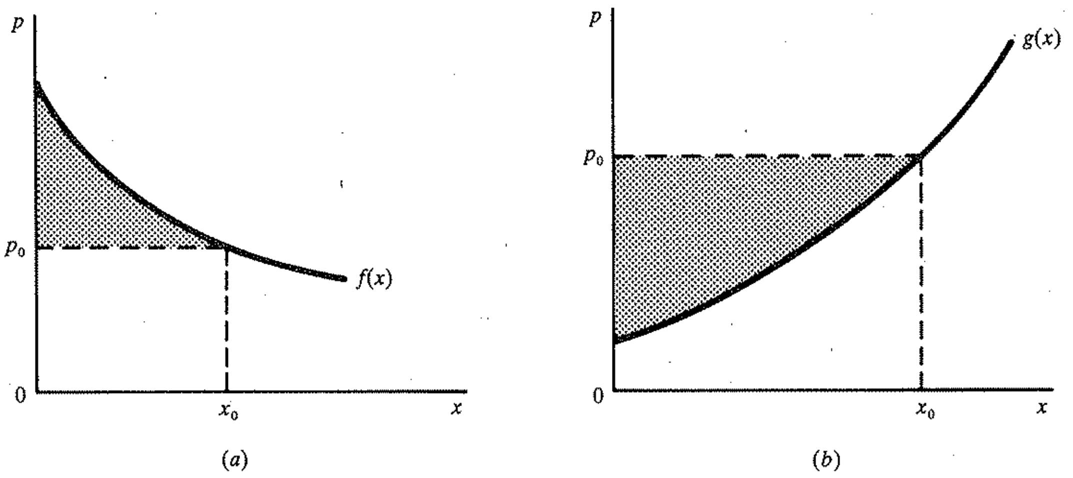


Fig. 8-6

EXAMPLE 12. A demand function $p_1 = f(x)$, as in Fig. 8-6(*a*), represents the different prices consumers are willing to pay for different quantities of a good. If equilibrium in the market occurs at (x_0, p_0) with all consumers paying the same price, the consumers who would have bought the good even at a higher price benefit. Total benefit to consumers, called *consumers' surplus*, is depicted by the shaded area. Mathematically,

$$\text{Consumers' surplus} = \int_0^{x_0} f(x)\,dx - p_0 x_0 \tag{8.19}$$

A supply function $p_2 = g(x)$, as in Fig. 8-6(*b*), represents the prices at which producers will supply different quantities of a good. If market equilibrium occurs at (x_0, p_0), producers willing to supply at lower prices than p_0 benefit. Total gain to producers is termed *producers' surplus* and is designated by the shaded area. Mathematically,

$$\text{Producers' surplus} = p_0 x_0 - \int_0^{x_0} g(x)\,dx \tag{8.20}$$

See Examples 13 and 14 and Problems 8.28–8.29.

EXAMPLE 13. Given the demand function $p = 70 - x^2$ and assuming that at market equilibrium $p_0 = 34$ and $x_0 = 6$, the consumers' surplus is estimated as follows, using (*8.19*):

$$\begin{aligned} \text{Consumers' surplus} &= \int_0^6 (70 - x^2)\,dx - (6)(34) \\ &= [70x - \tfrac{1}{3}x^3]_0^6 - 204 \\ &= (420 - 72) - (0) - 204 = 144 \end{aligned}$$

EXAMPLE 14. Given the supply function $p = (x+4)^2$ and assuming that at market equilibrium $p_0 = 81$ and $x_0 = 5$, the producers' surplus is estimated as follows, using (8.20):

$$\text{Producers' surplus} = (81)(5) - \int_0^5 (x+4)^2\,dx$$
$$= 405 - \left[\tfrac{1}{3}(x+4)^3\right]_0^5$$
$$= 405 - \tfrac{1}{3}[(9)^3 - (4)^3] = 183.33$$

For a variety of different applications, see Problems 8.16–8.34.

Solved Problems

INDEFINITE INTEGRALS

8.1 Find the following indefinite integrals. Check the answers on your own by making sure that the derivative of the antiderivative equals the integrand.

(a) $\int 7\,dx$ (b) $\int -12\,dx$ (c) $\int x^4\,dx$ (d) $\int \frac{8}{x^5}\,dx$

(e) $\int (20x^4 - 8x^3)\,dx$ (f) $\int \sqrt{x}\,dx$ (g) $\int 12e^{-3t}\,dt$ (h) $\int \frac{2}{x}\,dx$

(a) $$\int 7\,dx = 7x + c \qquad \text{[Rule 1]}$$

(b) $$\int -12\,dx = -\int 12\,dx = -12x + c \qquad \text{[Rule 8]}$$

(c) $$\int x^4\,dx = \frac{1}{4+1}x^{4+1} + c = \frac{1}{5}x^5 + c \qquad \text{[Rule 3]}$$

(d) $$\int \frac{8}{x^5}\,dx = 8\int x^{-5}\,dx = (8)\left(\frac{1}{-4}\right)x^{-4} + c = -2x^{-4} + c \qquad \text{[Rules 3 and 6]}$$

(e) $$\int (20x^4 - 8x^3)\,dx = \tfrac{20}{5}x^5 - \tfrac{8}{4}x^4 + c = 4x^5 - 2x^4 + c \qquad \text{[Rules 6, 7, and 8]}$$

(f) $$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{\frac{3}{2}}x^{3/2} + c = \frac{2}{3}x^{3/2} + c \qquad \text{[Rule 3]}$$

(g) $$\int 12e^{-3t}\,dt = 12\left(\frac{1}{-3}\right)e^{-3t} + c = -4e^{-3t} + c \qquad \text{[Rules 5 and 6]}$$

(h) $$\int \frac{2}{x}\,dx = \int 2x^{-1}\,dx = 2\ln|x| + c = \ln x^2 + c \qquad \text{[Rules 4 and 6]}$$

8.2 Determine the following indefinite integrals:

(a) $\int 10e^{t/5}\,dt$ (b) $\int x^{1/4}\,dx$ (c) $\int \frac{dx}{\sqrt{x}}$ (d) $\int \frac{8}{t^5}\,dt$

(e) $\int \sqrt{x+5}\,dx$ (f) $\int \frac{1}{x+9}\,dx$ (g) $\int 6(x-15)^{-2}\,dx$

(h) $\int (e^3 + 5t^4 + 3e^{-4t})\,dt$

(a) $$\int 10e^{t/5}\,dt = \int 10e^{(1/5)t}\,dt = 10\left(\frac{1}{\frac{1}{5}}\right)e^{(1/5)t} + c = 50e^{t/5} + c$$

(b) $$\int x^{1/4}\,dx = \frac{1}{\frac{5}{4}}x^{5/4} + c = \frac{4}{5}x^{5/4} + c$$

(2) $$A = \int_6^8 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_6^8 = \ln 9 - \ln 7$$
$$= 2.19722 - 1.94591 = .25131 \approx .25$$

(c) (1) $$A \approx \sum_{i=1}^{5} [(f(x_i)\,\Delta x_i]$$
$$\approx [e^{(1/4)x_1} + e^{(1/4)x_2} + e^{(1/4)x_3} + e^{(1/4)x_4} + e^{(1/4)x_5}](\Delta x)$$

Substituting the values for Δx and x_i from Problem 8.7(*c*),

$$A \approx [e^{(2.4)/4} + e^{(3.2)/4} + e^{(4)/4} + e^{(4.8)/4} + e^{(5.6)/4}](.8)$$
$$\approx (e^{.6} + e^{.8} + e^{1} + e^{1.2} + e^{1.4})(.8)$$
$$\approx (14.14126)(.8) \approx 11.31301$$

(2) $$A = \int_2^6 e^{x/4}\,dx = 4e^{x/4}\Big|_2^6 = 4e^{1.5} - 4e^{.5} = 17.92676 - 6.59489 = 11.33187$$

(d) (1) $$A \approx \sum_{i=1}^{3} [f(x_i)\,\Delta x_i] \approx [(x_1+5) + (x_2+5) + (x_3+5)](\Delta x)$$

Substituting the values found in Problem 8.7(*d*),

$$A \approx [(1.4+5) + (1.8+5) + (2.2+5)](.4) \approx 8.16$$

(2) $$A = \int_{1.2}^{2.4} (x+5)\,dx = (\tfrac{1}{2}x^2 + 5x)\Big|_{1.2}^{2.4}$$
$$= [\tfrac{1}{2}(2.4)^2 + 5(2.4)] - [\tfrac{1}{2}(1.2)^2 + 5(1.2)] = 8.16$$

APPLICATIONS

8.9 An object is catapulted upward from a launcher 352 feet above the ground with a velocity at time t given by $V(t) = 144 - 32t$. Find (*a*) the height S of the object at time t, (*b*) how long it will take the object to hit the ground, and (*c*) the maximum height the object will reach.

(*a*) Velocity is the rate of change in distance over time, that is, $V(t) = S'(t)$. Reversing the process,

$$S(t) = \int V(t)\,dt = \int (144 - 32t)\,dt$$
$$= 144t - 16t^2 + c \qquad (8.22)$$

At $t = 0$, $S = 352$. Substituting in (*8.22*),

$$352 = 144(0) - 16(0)^2 + c$$
$$c = 352 \qquad \text{and} \qquad S(t) = 144t - 16t^2 + 352$$

(*b*) When the object strikes the ground, $S(t) = 0$. Substituting 0 for S and solving for t,

$$S(t) = 144t - 16t^2 + 352 = 0$$
$$-16(t^2 - 9t - 22) = 0$$
$$(t - 11)(t + 2) = 0$$
$$t = 11 \text{ seconds}$$

(*c*) Maximizing $S(t) = -16t^2 + 144t + 352$,

$$S'(t) = -32t + 144 = 0 \qquad t = 4.5 \text{ seconds}$$

and $$S(4.5) = -16(4.5)^2 + 144(4.5) + 352 = 676 \text{ feet}$$

8.10 A stone is dropped from a height of 1024 feet and falls at a velocity $V(t) = -32t$ feet per second. Find (*a*) the height $S(t)$ of the stone for any time t, (*b*) how long it will take the stone to strike the ground, and (*c*) the velocity of the stone upon impact.

(a) $$S(t) = \int (-32t)\, dt = -16t^2 + c \qquad (8.23)$$

At $t = 0$, $S = 1024$. Substituting in (8.23),

$$1024 = -16(0)^2 + c \qquad c = 1024$$
$$S(t) = -16t^2 + 1024$$

(b) When the stone hits the ground, $S(t) = 0$.

$$S(t) = -16t^2 + 1024 = 0$$
$$t^2 = 64 \qquad t = 8 \text{ seconds}$$

(c) The velocity at $t = 8$ is

$$V(8) = -32(8) = -256 \text{ feet per second}$$

8.11 Find the average value m of the following functions on the interval $[a, b]$:

(a) $f(x) = \dfrac{1}{\sqrt{x+3}}$; $a = 1$, $b = 6$ (b) $f(x) = e^{x/6}$ from $x = 0$ to $x = 6$

(c) $f(x) = 2x + 8$; $a = 3$, $b = 7$ (d) $f(x) = \sqrt{x-2}$; $a = 6$, $b = 11$

(a) From (8.17), $$m = \frac{1}{b-a}\int_a^b f(x)\, dx$$

Substituting, $$m = \frac{1}{6-1}\int_1^6 \frac{1}{\sqrt{x+3}}\, dx = \frac{1}{5}\int_1^6 (x+3)^{-1/2}\, dx$$
$$= \tfrac{1}{5}[2(x+3)^{1/2}]\Big|_1^6 = \tfrac{2}{5}(\sqrt{x+3})\Big|_1^6$$
$$= \tfrac{2}{5}(\sqrt{9} - \sqrt{4}) = \tfrac{2}{5}(3-2) = \tfrac{2}{5}$$

(b) From (8.17), $$m = \frac{1}{6-0}\int_0^6 e^{x/6}\, dx$$
$$= \tfrac{1}{6}(6e^{x/6})\Big|_0^6 = e^{(1)} - e^{(0)} = e - 1 = 1.71828$$

(c) $$m = \frac{1}{7-3}\int_3^7 (2x+8)\, dx = \frac{1}{4}(x^2+8x)\Big|_3^7$$
$$= \tfrac{1}{4}[(49+56) - (9+24)] = 18$$

(d) $$m = \frac{1}{11-6}\int_6^{11} (x-2)^{1/2}\, dx = \frac{1}{5}\left[\frac{2}{3}(x-2)^{3/2}\right]\Bigg|_6^{11}$$
$$= \tfrac{2}{15}[(9)^{3/2} - (4)^{3/2}] = \tfrac{2}{15}(27-8) = 2\tfrac{8}{15}$$

8.12 (1) Find the volume V of the solid of revolution generated by revolving around the x-axis the regions under each of the following curves. (2) Draw a rough sketch of the graphs.

(a) $f(x) = 5x^2$; $a = 1$, $b = 3$ (b) $f(x) = 3x + 2$; $a = 0$, $b = 4$

(c) $f(x) = \sqrt{9 - x^2}$; $a = -3$, $b = 3$ (d) $f(x) = e^{-1.5x}$; $a = 0$, $b = 1$

(a) (1) From (8.18), $$V = \int_a^b \pi[f(x)]^2\, dx$$

Substituting, $$V = \int_1^3 \pi(5x^2)^2\, dx$$

Squaring and rearranging constants,

$$V = 25\pi\int_1^3 x^4\, dx = 25\pi[\tfrac{1}{5}x^5]_1^3 = 5\pi(243 - 1) = 1210\pi$$

(2) See Fig. 8-16.

8.14 Revolution about the x-axis of the area under the curve $y = (r/h)x$ from $x = 0$ to $x = h$ generates a cone of height h and radius r. Derive the formula for the volume of a cone.

$$V = \int_0^h \pi\left[\left(\frac{r}{h}\right)x\right]^2 dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi r^2}{h^2}\left[\tfrac{1}{3}x^3\right]_0^h = \tfrac{1}{3}\pi r^2 h$$

8.15 Rotation around the x-axis of the region under the curve $y = \sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$ generates a sphere of radius r. Derive the formula for the volume of a sphere of radius r.

$$\begin{aligned} V &= \int_{-r}^{r} \pi[(r^2 - x^2)^{1/2}]^2\,dx \\ &= \pi\int_{-r}^{r}(r^2 - x^2)\,dx = \pi[r^2x - \tfrac{1}{3}x^3]_{-r}^{r} \\ &= \pi[(r^3 - \tfrac{1}{3}r^3) - (-r^3 + \tfrac{1}{3}r^3)] = \tfrac{4}{3}\pi r^3 \end{aligned}$$

PRACTICAL APPLICATIONS

8.16 A pipe on an offshore drilling platform is damaged, spilling oil at a rate of $(35t + 80)$ barrels per hour t. How many barrels will be leaked the first day?

$$\begin{aligned} B(t) &= \int_0^{24}(35t + 80)\,dt = (17.5t^2 + 80t)\Big|_0^{24} \\ &= 10\,080 + 1920 = 12\,000 \end{aligned}$$

8.17 Dendrologists have estimated that a particular tree grows at a rate of $[2.5 + 1/(t+2)^2]$ feet per year t. How much will it grow in the third year?

$$\begin{aligned} G(t) &= \int_2^3\left[2.5 + \frac{1}{(t+2)^2}\right]dt = \int_2^3[2.5 + (t+2)^{-2}]\,dt \\ &= [2.5t - (t+2)^{-1}]_2^3 = \left(2.5t - \frac{1}{t+2}\right)\Big|_2^3 \\ &= 2.55 \text{ feet in the third year} \end{aligned}$$

8.18 A person's rate of reaction or sensitivity to a specific drug t hours after it is administered is given by

$$S'(t) = \frac{3}{t} + \frac{4}{t^2}$$

where S is measured in suitable units. Find the strength of the total reaction from $t = 1$ to $t = 8$.

$$S(t) = \int_1^8(3t^{-1} + 4t^{-2})\,dt = \left(3\ln t - \frac{4}{t}\right)\Big|_1^8 \approx 9.7$$

8.19 A firm's marginal cost function is $C'(x) = x^2 - 4x + 110$, with x representing the number of units per day. Fixed costs are \$340 a day. What is the total cost $C(x)$ of producing x units per day?

$$C(x) = \int(x^2 - 4x + 110)\,dx = \tfrac{1}{3}x^3 - 2x^2 + 110x + c$$

Substituting $C(0) = 340$,

$$340 = \tfrac{1}{3}(0)^3 - 2(0)^2 + 110(0) + c \qquad c = 340$$

and

$$C(x) = \tfrac{1}{3}x^3 - 2x^2 + 110x + 340$$

8.20 A producer's marginal cost is $C'(x) = \frac{1}{12}x^2 - x + 180$. What is the total cost $C(x)$ of producing five extra units if three units are currently being produced?

$$C(8) - C(3) = \int_3^8 (\tfrac{1}{12}x^2 - x + 180)\,dx$$
$$= (\tfrac{1}{36}x^3 - \tfrac{1}{2}x^2 + 180x)\Big|_3^8 = 885.97$$

8.21 A manufacturer's marginal profit is $\pi' = -3x^2 + 80x + 140$. Find the profit π earned by increasing production from two units to four units.

$$\pi(4) - \pi(2) = \int_2^4 (-3x^2 + 80x + 140)\,dx$$
$$= (-x^3 + 40x^2 + 140x)\Big|_2^4 = 704$$

8.22 Maintenance costs $M(t)$ in a factory increase as plant and equipment get older. If the rate of increase in maintenance costs in dollars per year is $M'(t) = 75t^2 + 9000$, where t is years, find the total maintenance costs of the factory from year 4 to year 6.

$$M(6) - M(4) = \int_4^6 (75t^2 + 9000)\,dt$$
$$= (25t^3 + 9000t)\Big|_4^6 = 21\,800$$

8.23 A car depreciates rapidly in value in its first few years and more slowly in later years. Given the rate at which the value of a car depreciates over the years, $V'(t) = 300(t - 8)$ for $0 \le t \le 8$, and a sticker price of \$12 000, find (*a*) the value of the car $V(t)$, (*b*) the total amount by which the car depreciates in the first 4 years, and (*c*) the total amount by which it depreciates in the next 4 years. (*d*) Use the answer in part (*a*) evaluated at $t = 4$ to check your answer in part (*b*).

(*a*)
$$V(t) = \int 300(t - 8)\,dt = 150t^2 - 2400t + c$$

With $V(0) = 12\,000$,

$$V(t) = 150t^2 - 2400t + 12\,000 \qquad (8.24)$$

(*b*)
$$V(4) - V(0) = \int_0^4 300(t - 8)\,dt$$
$$= (150t^2 - 2400t)\Big|_0^4 = 2400 - 9600 = -7200$$

The value of the car decreases, that is, depreciates, by \$7200 in the first 4 years.

(*c*)
$$V(8) - V(4) = \int_4^8 300(t - 8)\,dt$$
$$= (150t^2 - 2400t)\Big|_4^8 = -2400$$

The car depreciates by \$2400 in the next 4 years.

(*d*) Evaluating (*8.24*) at $t = 4$,

$$V(4) = 150(4)^2 - 2400(4) + 12\,000 = 4800$$
$$V(0) - V(4) = 12\,000 - 4800 = 7200$$

The car has depreciated by \$7200 as was found in part (*b*).

8.24 The annual rate of water consumption in billions of gallons for a given community is $W'(t) = t + e^{0.02t}$, where $t = 0$ represents 1980. Find the total level of water consumption $W(t)$ for the period 1980–1990.

$$W(t) = \int_0^{10} (t + e^{.02t})\, dt$$

$$= \left(\frac{1}{2}t^2 + \frac{1}{.02}e^{.02t}\right)\Big|_0^{10} = (50 + 50e^{.2}) - (0 + 50)$$

$$= 50(1.22140) \approx 61 \text{ billion gallons}$$

8.25 A zinc mine extracts ore at a rate of thousands of tons per year, $Z'(t) = 18t - (22/\sqrt{t})$. Find the total amount of ore extracted from (*a*) year 0 to year 9, (*b*) year 0 to year *n*.

(*a*)
$$Z(t) = \int_0^9 (18t - 22t^{-1/2})\, dt = (9t^2 - 44t^{1/2})|_0^9 = 597\,000 \text{ tons}$$

(*b*)
$$Z(t) = \int_0^n (18t - 22t^{-1/2})\, dt = (9t^2 - 44t^{1/2})|_0^n = 9n^2 - 44\sqrt{n}$$

8.26 A gold mine extracts ore at a rate in tons of $G'(t) = 4.8e^{.016t}$. Find total extraction (*a*) from year 0 to year 10 and (*b*) from year 0 to year *n*.

(*a*)
$$G(t) = \int_0^{10} 4.8e^{.016t}\, dt = 300e^{.016t}|_0^{10}$$

$$= 300(e^{.16} - e^0) = 300(1.17351 - 1) \approx 52 \text{ tons}$$

(*b*)
$$G(t) = \int_0^n 4.8e^{.016t}\, dt = 300e^{.016t}|_0^n = 300(e^{.016n} - 1)$$

8.27 An oil company is pumping oil from its Alaskan field at an annual rate in billions of barrels given by $B'(t) = 1.6e^{.05t}$. (*a*) At this rate how much will it pump from year 0 to year 5? (*b*) If the field has a reserve of 16 billion barrels, in how many years *n* will the field run dry?

(*a*)
$$B(5) - B(0) = \int_0^5 1.6e^{.05t}\, dt = (32e^{.05t})|_0^5$$

$$= 32(e^{.25} - e^0) = 32(1.28403 - 1) \approx 9.1 \text{ bb}$$

(*b*)
$$\int_0^n 1.6e^{.05t}\, dt = 16$$

$$32e^{.05t}|_0^n = 32(e^{.05n} - 1) = 16$$

$$e^{.05t} - 1 = \tfrac{16}{32} = .5$$

$$e^{.05n} = 1.5$$

Taking the natural log of both sides, as in (*7.2*),

$$.05n = \ln 1.5 = .40547$$

$$n = \frac{.40547}{.05} \approx 8.1 \text{ years}$$

8.28 Find the consumers' surplus for each of the following demand curves at the level indicated:

(*a*) $p = 375 - 3x^2$; $x_0 = 10$, $p_0 = 75$ (*b*) $p = \dfrac{350}{x+5}$; $x_0 = 20$, $p_0 = 14$

(a) From (8.19),

$$\begin{aligned} CS &= \int_0^{x_0} f(x)\,dx - p_0x_0 \\ &= \int_0^{10} (375 - 3x^2)\,dx - 75(10) \\ &= (375x - x^3)\Big|_0^{10} - 750 \\ &= 3750 - 1000 - 750 = 2000 \end{aligned}$$

(b)

$$\begin{aligned} CS &= \int_0^{20} [350(x+5)^{-1}]\,dx - 14(20) \\ &= [350 \ln (x+5)]_0^{20} - 280 \\ &= [350(3.21888 - 1.60944)] - 280 \approx 283.30 \end{aligned}$$

8.29 Find the producers' surplus for each of the following supply curves at the level indicated:

(a) $p = x^2 + 4x + 60;\ x_0 = 5,\ p_0 = 85$ (b) $p = 5 + \frac{1}{4}\sqrt{x};\ x_0 = 144,\ p_0 = 8$

(a) From (8.20),

$$\begin{aligned} PS &= p_0x_0 - \int_0^{x_0} g(x)\,dx \\ &= 85(5) - \int_0^{5} (x^2 + 4x + 60)\,dx \\ &= 425 - (\tfrac{1}{3}x^3 + 2x^2 + 60x)\Big|_0^{5} = 425 - 391.67 = 33.33 \end{aligned}$$

(b)

$$\begin{aligned} PS &= 8(144) - \int_0^{144} (5 + \tfrac{1}{4}x^{1/2})\,dx \\ &= 1152 - [5x + \tfrac{1}{6}x^{3/2}]_0^{144} = 1152 - 720 - 288 = 144 \end{aligned}$$

8.30 A manufacturer introduces a new technique that brings about a rate of savings in dollars per year $S'(t) = 400 - t^2$. The marginal cost of production in dollars per year also increases because of the new technique and is given by $C'(t) = t^2 + 20t$. (a) Determine how long the use of the new technique will prove profitable and (b) the total amount T saved during this period. A sketch of the graphs, as in Fig. 8-20, would help.

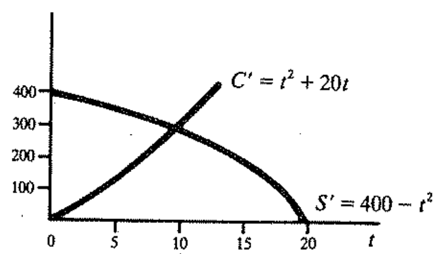


Fig. 8-20

(a) The new technique is profitable until the two curves intersect. Setting $S'(t)$ equal to $C'(t)$ and solving for t,

$$\begin{aligned} 400 - t^2 &= t^2 + 20t \\ 2(t^2 + 10t - 200) &= 0 \\ (t+20)(t-10) &= 0 \qquad t = 10 \end{aligned}$$

(*b*)
$$T = \int_0^{10} [S'(t) - C'(t)]\, dt$$
$$= \int_0^{10} [(400 - t^2) - (t^2 + 20t)]\, dt = \int_0^{10} (400 - 20t - 2t^2)\, dt$$
$$= [400t - 10t^2 - \tfrac{2}{3}t^3]_0^{10} = 2333.33$$

8.31 According to engineers' estimates, the cost of a new product is $C = 6\sqrt{x} + 15$. Find the average cost m of producing the first 64 units.

From (*8.17*),
$$m = \frac{1}{b-a}\int_a^b f(x)\, dx = \frac{1}{64-0}\int_0^{64} (6x^{1/2} + 15)\, dx$$
$$= \tfrac{1}{64}[4x^{3/2} + 15x]_0^{64} = 47$$

8.32 A firm's inventory after t months is given by $N(t) = 25 + 48t - 4t^2$ for $0 \le t \le 12$. Find the average inventory m during the first quarter of the year, that is, the first 3 months.
$$m = \frac{1}{3-0}\int_0^3 (25 + 48t - 4t^2)\, dt$$
$$= \tfrac{1}{3}[25t + 24t^2 - \tfrac{4}{3}t^3]_0^3 = \tfrac{1}{3}(75 + 216 - 36) = 85$$

8.33 The population in millions of people for a newly emerging nation is $P(t) = 18e^{.032t}$. Find the average population m over the next 25 years.
$$m = \frac{1}{25-0}\int_0^{25} 18e^{.032t}\, dt$$
$$= \tfrac{1}{25}[562.5e^{.032t}]_0^{25} = 22.5(e^{.8} - 1) \approx 27.6 \text{ million}$$

8.34 Ten thousand dollars is deposited in a bank at 8% interest compounded continuously. Find the average value of the money over the next 5 years.

From (*7.10*),
$$A(t) = 10\,000e^{.08t}$$

Hence,
$$m = \frac{1}{5-0}\int_0^5 10\,000e^{.08t}\, dt$$
$$= \tfrac{1}{5}[125\,000e^{.08t}]_0^5 = 25\,000(e^{.4} - 1) \approx 12\,296$$