Chapter 5

Differentiation

5.1 DERIVATIVE NOTATION

The derivative of a function can be expressed in several different ways, with many different symbols, including the Greek alphabet. If $y = f(x)$, the derivative can be expressed as

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}[f(x)] \qquad \text{or} \qquad D_x[f(x)]$$

If $y = \phi(t)$, the derivative can be written

$$\phi'(t) \qquad y' \qquad \frac{dy}{dt} \qquad \frac{d\phi}{dt} \qquad \frac{d}{dt}[\phi(t)] \qquad \text{or} \qquad D_t[\phi(t)]$$

If the derivative of $y = f(x)$ is evaluated at $x = a$, proper notation includes $f'(a)$ and $\left.\frac{dy}{dx}\right|_a$.

EXAMPLE 1. If $y = 5x^2 + 7x + 12$, the derivative can be written

$$y' \qquad \frac{dy}{dx} \qquad \frac{d}{dx}(5x^2 + 7x + 12) \qquad \text{or} \qquad D_x(5x^2 + 7x + 12)$$

If $z = \sqrt{8t - 3}$, the derivative can be expressed as

$$z' \qquad \frac{dz}{dt} \qquad \frac{d}{dt}(\sqrt{8t - 3}) \qquad \text{or} \qquad D_t(\sqrt{8t - 3})$$

See Problems 5.1–5.2.

5.2 RULES OF DIFFERENTIATION

Differentiation refers to the process of finding the derivative of a function. In explaining the rules of differentiation for a function $y = f(x)$, other functions such as $g(x)$ and $h(x)$ are frequently used, where g and h are both unspecified functions of x. The rules of differentiation are cataloged below with illustrations and treated in Problems 5.3–5.15. A sampling of proofs is handled in Section 5.3.

5.2.1 The Constant Function Rule

The derivative of a constant function $f(x) = k$, where k is a constant, is zero.

$$\text{Given } f(x) = k, \qquad f'(x) = 0$$

EXAMPLE 2.

$$\text{Given } f(x) = 7, \qquad f'(x) = 0$$
$$\text{Given } f(x) = -2, \qquad f'(x) = 0$$

5.2.2 The Linear Function Rule

The derivative of a linear function $f(x) = mx + b$ is equal to m, the coefficient of x. The derivative of a variable raised to the first power is always equal to the coefficient of the variable, while the derivative of a constant is simply zero.

$$\text{Given } f(x) = mx + b, \qquad f'(x) = m$$

EXAMPLE 3.

$$\text{Given } f(x) = 4x + 9, \qquad f'(x) = 4$$
$$\text{Given } f(x) = \tfrac{1}{2}x + 3, \qquad f'(x) = \tfrac{1}{2}$$
$$\text{Given } f(x) = -16x, \qquad f'(x) = -16$$

5.2.3 The Power Function Rule

The derivative of a power function $f(x) = x^n$, where n is any real number, is equ exponent n multiplied by the variable x raised to the $(n-1)$ power.

$$\text{Given } f(x) = x^n, \qquad f'(x) = nx^{n-1}$$

EXAMPLE 4.

$$\text{Given } f(x) = x^3, \qquad f'(x) = 3x^{3-1} = 3x^2$$
$$\text{Given } f(x) = x^5, \qquad f'(x) = 5x^{5-1} = 5x^4$$
$$\text{Given } f(x) = x^2, \qquad f'(x) = 2x^{2-1} = 2x$$

See also Problem 5.4.

5.2.4 The Rule for a Constant Times a Function

The derivative of a constant times a function $f(x) = kg(x)$, where k is a real number anc differentiable function, is equal to the constant k times the derivative of the function $g'(x$

$$\text{Given } f(x) = kg(x), \qquad f'(x) = kg'(x)$$

EXAMPLE 5.

$$\text{Given } f(x) = 6x^4, \qquad f'(x) = 6(4x^{4-1}) = 24x^3$$
$$\text{Given } f(x) = -7x^6, \qquad f'(x) = -7(6x^{6-1}) = -42x^5$$

For the derivation of this rule, see Problem 5.16.

5.2.5 The Rule for Sums and Differences

The derivative of a sum of two functions $f(x) = g(x) + h(x)$, where $g(x)$ and $h(x)$ differentiable functions, is equal to the sum of the derivatives of the individual functions. S the derivative of the difference of two functions is equal to the difference of the derivativ two functions.

$$\text{Given } f(x) = g(x) \pm h(x), \qquad f'(x) = g'(x) \pm h'(x)$$

EXAMPLE 6.

$$\text{Given } f(x) = 8x^4 + 5x^3, \qquad f'(x) = 32x^3 + 15x^2$$
$$\text{Given } f(x) = 3x^2 - 7x - 2, \qquad f'(x) = 6x - 7$$

See also Problem 5.5. For the derivation see Example 12.

5.2.6 The Product Rule

The derivative of a product $f(x) = g(x) \cdot h(x)$, where $g(x)$ and $h(x)$ are both diffe functions, is equal to the first function multiplied by the derivative of the second function second function multiplied by the derivative of the first function. Given $f(x) = g(x) \cdot h(x)$,

$$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

EXAMPLE 7. Given $f(x) = 8x^3(5x-2)$, let $g(x) = 8x^3$ and $h(x) = 5x - 2$. Taking the individual de $g'(x) = 24x^2$ and $h'(x) = 5$. Then substituting these values in the product rule formula (*5.1*),

$$f'(x) = 8x^3(5) + (5x-2)(24x^2)$$

and simplifying algebraically,

$$f'(x) = 40x^3 + 120x^3 - 48x^2 = 160x^3 - 48x^2$$

See also Problems 5.6–5.8; for the derivation of the rule, see Problem 5.17.

5.2.7 The Quotient Rule

The derivative of a quotient $f(x) = g(x) \div h(x)$, where $g(x)$ and $h(x)$ are both differ functions and $h(x) \neq 0$, is equal to the denominator times the derivative of the numerator, m numerator times the derivative of the denominator, all divided by the denominator squared. Given $f(x) = g(x)/h(x)$,

$$f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2} \tag{5.2}$$

EXAMPLE 8. Given

$$f(x) = \frac{2x^4}{5x - 6}$$

where $g(x) = 2x^4$ and $h(x) = 5x - 6$; $g'(x) = 8x^3$ and $h'(x) = 5$. Substituting these values in the quotient rule formula (*5.2*),

$$f'(x) = \frac{(5x-6)(8x^3) - 2x^4(5)}{(5x-6)^2}$$

Simplifying algebraically,

$$f'(x) = \frac{40x^4 - 48x^3 - 10x^4}{(5x-6)^2} = \frac{30x^4 - 48x^3}{(5x-6)^2} = \frac{6x^3(5x-8)}{(5x-6)^2}$$

See also Problems 5.9–5.10; for the derivation of the rule, see Problem 5.18.

5.2.8 The Generalized Power Function Rule

The derivative of a function raised to a power $f(x) = [g(x)]^n$, where $g(x)$ is a differentiable function and n is any real number, is equal to the exponent n times the function $g(x)$ raised to the $(n-1)$ power, multiplied in turn by the derivative of the function itself $g'(x)$. Given $f(x) = [g(x)]^n$,

$$f'(x) = n[g(x)]^{n-1} \cdot g'(x) \tag{5.3}$$

Note: While the generalized power function rule is derived from the chain rule which follows, it is presented first because it is usually easier for students to understand.

EXAMPLE 9. Given $f(x) = (x^2 + 7)^4$, let $g(x) = x^2 + 7$, then $g'(x) = 2x$. Substituting these values in the generalized power function formula (*5.3*),

$$f'(x) = 4(x^2 + 7)^{4-1} \cdot 2x$$

Simplifying algebraically,

$$f'(x) = 4(x^2 + 7)^3 \cdot 2x = 8x(x^2 + 7)^3$$

See Problems 5.11–5.12.

5.2.9 The Chain Rule

The derivative of a composite function $f(x) = g[h(x)]$, where $g(x)$ and $h(x)$ are both differentiable functions, is equal to the derivative of the first function $g'(x)$, in which $h(x)$ is substituted for x, times the derivative of the second function $h'(x)$. Given $f(x) = g[h(x)]$,

$$f'(x) = g'[h(x)] \cdot h'(x) \tag{5.4}$$

The chain rule is also called the *composite function rule* or the *function of a function rule*. For composite functions, see Section 3.3 and Problem 3.11.

EXAMPLE 10. Given $f(x) = \sqrt{8x + 9} = (8x + 9)^{1/2}$ let

$$g(x) = x^{1/2} \quad \text{and} \quad h(x) = 8x + 9$$

then

$$g'(x) = \tfrac{1}{2}x^{-1/2} \quad \text{and} \quad h'(x) = 8$$

Substituting $h(x)$ for x in $g'(x)$ in preparation to using the chain rule,

$$g'[h(x)] = \tfrac{1}{2}(8x + 9)^{-1/2}$$

Finally, substituting the appropriate values into the chain rule formula (*5.4*),

$$f'(x) = \frac{1}{2}(8x+9)^{-1/2} \cdot 8 = 4(8x+9)^{-1/2} = \frac{4}{\sqrt{8x+9}}$$

This function could also have been solved with the generalized power function rule, which is : specialized use of the chain rule. See Problems 5.13–5.14.

EXAMPLE 11. The chain rule is also expressed in the following notation which is worth noting. : function of u and u in turn is a function of x, that is, $y = f(u)$ and $u = g(x)$, then the derivative of y wit to x is given by the chain rule as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Given $y = (x^2 + 5)^3$, and purposely using the chain rule from (*5.5*) instead of the generalized power rule, let

$$y = u^3 \qquad \text{and} \qquad u = x^2 + 5$$

then

$$\frac{dy}{du} = 3u^2 \qquad \text{and} \qquad \frac{du}{dx} = 2x$$

Substituting these values in the new chain rule formula (*5.5*),

$$\frac{dy}{dx} = 3u^2 \cdot 2x = 6xu^2$$

Finally, substituting $u = x^2 + 5$,

$$\frac{dy}{dx} = 6x(x^2+5)^2$$

as could also be found with the generalized power function rule.

5.3 DERIVATION OF THE RULES OF DIFFERENTIATION

The rules of differentiation are derived from the definition of the derivative in (*4.4*). B illustration, the rule for finding the derivative of differences is derived below in Example 1 important rules are derived in Problems 5.16–5.18.

EXAMPLE 12. Given $f(x) = g(x) - h(x)$ and assuming $g'(x)$ and $h'(x)$ both exist, then from (derivative of $f(x)$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Substituting $f(x) = g(x) - h(x)$,

$$f'(x) = \lim_{\Delta x \to 0} \frac{[g(x+\Delta x) - h(x+\Delta x)] - [g(x) - h(x)]}{\Delta x}$$

First rearranging terms, then factoring out a negative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x) - h(x+\Delta x) + h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x) - [h(x+\Delta x) - h(x)]}{\Delta x}$$

Now separating terms and limits,

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} - \frac{h(x+\Delta x) - h(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} - \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

$$= g'(x) - h'(x)$$

5.4 HIGHER-ORDER DERIVATIVES

The second-order derivative, written $f''(x)$, measures the slope and the rate of change of the first derivative, just as the first derivative measures the slope and the rate of change of the original or *primitive function*. The third-order derivative $[f'''(x)]$ measures the slope and rate of change of the second-order derivative, etc. Higher-order derivatives are found by applying the rules of differentiation to lower-order derivatives, as illustrated in Example 13 and treated in Problems 5.19–5.20.

5.5 HIGHER-ORDER DERIVATIVE NOTATION

Given $y = f(x)$, commonly used higher-order derivative notation includes:

First order:	$f'(x)$	y'	$\dfrac{dy}{dx}$	$\dfrac{df}{dx}$	$\dfrac{d}{dx}(y)$	$D_x[f(x)]$
Second order:	$f''(x)$	y''	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^2f}{dx^2}$	$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$	$D_x^2[f(x)]$
Third order:	$f'''(x)$	y'''	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^3f}{dx^3}$	$\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$	$D_x^3[f(x)]$
Fourth order:	$f^{(4)}(x)$	$y^{(4)}$	$\dfrac{d^4y}{dx^4}$	$\dfrac{d^4f}{dx^4}$	$\dfrac{d}{dx}\left(\dfrac{d^3y}{dx^3}\right)$	$D_x^4[f(x)]$
*n*th order:	$f^{(n)}(x)$	$y^{(n)}$	$\dfrac{d^ny}{dx^n}$	$\dfrac{d^nf}{dx^n}$	$\dfrac{d}{dx}\left[\dfrac{d^{(n-1)}y}{dx^{(n-1)}}\right]$	$D_x^n[f(x)]$

Note: (1) The position of the superscripts in $\dfrac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{(dx)^2}$$

(2) The parentheses in the fourth-order derivative $y^{(4)}$

$$y^{(4)} \neq y^4 = y \cdot y \cdot y \cdot y$$

EXAMPLE 13. Higher-order derivatives are found by applying the rules of differentiation to the derivative of the previous order. Given $f(x) = 4x^3 - 9x^2 + 5x - 7$, then

$$f'(x) = 12x^2 - 18x + 5$$
$$f''(x) = 24x - 18$$
$$f'''(x) = 24$$
$$f^{(4)}(x) = 0$$

5.6 IMPLICIT DIFFERENTIATION

The functions we have encountered so far have generally been *explicit functions* with y to the left of the equal sign and all the x terms to the right. Functions in which x and y are both located on the same side of the equal sign are called *implicit functions*. Some implicit functions can be converted easily to explicit functions by solving for y in terms of x; others cannot.

For implicit functions and equations which cannot readily be solved for y in terms of x, the derivative dy/dx may be found by means of *implicit differentiation*, explained in Examples 15 and 16 and treated in Problems 5.21–5.23.

EXAMPLE 14. Given below are examples of explicit and implicit functions.

Explicit: $y = 4x,\quad y = x^2 + 6x - 7,\quad y = (x^4 - 9x^3)/(x^2 -$

Implicit: $8x + 5y = 21,\quad 3x^2 - 8xy - 5y = 49,\quad 35x^3y = 10$

EXAMPLE 15. Given the equation $4x^2 + 5y^3 = 39$, the derivative dy/dx can be found differentiation in two easy steps:

(1) Differentiate both sides of the equation with respect to x while treating y as a

$$\frac{d}{dx}(4x^2 + 5y^3) = \frac{d}{dx}(39)$$

$$\frac{d}{dx}(4x^2) + \frac{d}{dx}(5y^3) = \frac{d}{dx}(39)$$

where $\frac{d}{dx}(4x^2) = 8x$ and $\frac{d}{dx}(39) = 0$. Using the generalized power function rule noting that $\frac{d}{dx}(y) = \frac{dy}{dx}$,

$$\frac{d}{dx}(5y^3) = 5 \cdot 3 \cdot y^{3-1} \cdot \frac{d}{dx}(y) = 15y^2\frac{dy}{dx}$$

Substituting these values in (5.6),

$$8x + 15y^2\frac{dy}{dx} = 0$$

(2) Now simply solve (5.7) algebraically for $\frac{dy}{dx}$,

$$15y^2\frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = \frac{-8x}{15y^2}$$

EXAMPLE 16. Given the equation $x^3y^5 = 9$, implicit differentiation is used to find the deriv similar manner to that used above.

(1) Differentiating both sides of the equation with respect to x,

$$\frac{d}{dx}(x^3y^5) = \frac{d}{dx}(9)$$

and using the product rule and the generalized power function rule, as above, because a function of x,

$$x^3 \cdot \frac{d}{dx}(y^5) + y^5 \cdot \frac{d}{dx}(x^3) = \frac{d}{dx}(9)$$

$$x^3 \cdot 5y^4\frac{dy}{dx} + y^5 \cdot 3x^2 = 0$$

(2) Then solving for dy/dx algebraically,

$$5x^3y^4\frac{dy}{dx} = -3x^2y^5$$

$$\frac{dy}{dx} = \frac{-3x^2y^5}{5x^3y^4} = \frac{-3y}{5x}$$

5.7 APPLICATIONS FOR BUSINESS, ECONOMICS, AND THE SOCIAL SCIENCES

Earlier we saw that rates of change measured by the derivative are important to scien fields. With the rules of differentiation now in our arsenal, we are in a better position to more complicated and realistic functions. See Examples 17–19 and Problems 5.24–5.38.

EXAMPLE 17. With a total cost function $C = .001x^3 - .05x^2 + 7x + 6500$, (*a*) the marginal cost fu (*b*) the marginal cost of the fortieth unit are

(a) $$\text{MC} = \frac{dC}{dx} = .001(3x^2) - .05(2x) + 7 = .003x^2 - .1x + 7$$

(b) $$\text{MC}(40) = \left.\frac{dC}{dx}\right|_{x=40} = .003(40)^2 - .1(40) + 7 = 7.8$$

EXAMPLE 18. For a total revenue function,

$$R = \frac{600x}{x+3}$$

(a) the marginal revenue function and (b) the marginal revenue of the seventeenth unit are, by means of the quotient rule,

(a) $$\text{MR} = \frac{dR}{dx} = \frac{(x+3)600 - 600x(1)}{(x+3)^2}$$

$$= \frac{600x + 1800 - 600x}{(x+3)^2} = \frac{1800}{(x+3)^2}$$

(b) $$\text{MR}(17) = \left.\frac{dR}{dx}\right|_{x=17} = \frac{1800}{[(17)+3]^2} = \frac{1800}{400} = 4.5$$

EXAMPLE 19. Given the amount of medication in milligrams per cubic centimeter remaining in the bloodstream after t hours

$$M(t) = \frac{4}{\sqrt{3t+1}} = 4(3t+1)^{-1/2}$$

(a) the rate of change of M and (b) the rate of change of M at $t = 5$ are, via the generalized power function rule,

(a) $$M'(t) = 4 \cdot (-\tfrac{1}{2}) \cdot (3t+1)^{-3/2} \cdot 3 = -6(3t+1)^{-3/2} = \frac{-6}{\sqrt{(3t+1)^3}}$$

(b) $$M'(5) = \frac{-6}{\sqrt{[3(5)+1]^3}} = \frac{-6}{(16)^{3/2}} = \frac{-6}{64} \approx -0.09$$

Solved Problems

DERIVATIVE NOTATION

5.1 Using $u(x)$ and $v(x)$ as auxiliary functions, express the rules of differentiation asked for below in terms of dy/dx.

(a) The rule of sums (b) The product rule (c) The power function rule
(d) The quotient rule (e) The generalized power function rule

(a) Given $y = u(x) + v(x)$, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

(b) Given $y = u(x) \cdot v(x)$, $\dfrac{dy}{dx} = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$.

(c) Given $y = x^n$, $\dfrac{dy}{dx} = nx^{n-1}$.

(d) Given $y = \dfrac{u(x)}{v(x)}$, $\dfrac{dy}{dx} = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$

(e) Given $y = [u(x)]^n$, $\dfrac{dy}{dx} = n[u(x)]^{n-1} \cdot \dfrac{du}{dx}$.

5.2 Using $u(x)$ and $v(x)$ as auxiliary functions, express the rules of differentiation in terms of y' and D_x.

(*a*) The rule for differences (*b*) The product rule (*c*) The qu

(*a*) Given $y = u(x) - v(x)$,

$$y' = u' - v'$$
$$D_x[u(x) - v(x)] = D_x[u(x)] - D_x[v(x)]$$

(*b*) Given $y = u(x) \cdot v(x)$,

$$y' = uv' + vu'$$
$$D_x[u(x) \cdot v(x)] = u(x) \cdot D_x[v(x)] + v(x) \cdot D_x[u(x)]$$

(*c*) Given $y = u(x)/v(x)$,

$$y' = \frac{vu' - uv'}{v^2}$$
$$D_x\left[\frac{u(x)}{v(x)}\right] = \frac{v(x) \cdot D_x[u(x)] - u(x) \cdot D_x[v(x)]}{[v(x)]^2}$$

SIMPLE DERIVATIVES

5.3 Differentiate each of the following functions and practice the use of the differen a derivative:

(*a*) $f(x) = 13$ (*b*) $y = -27$ (*c*) $y = 7x - 12$ (*d*) $f(x) = 25$

(*a*) $f'(x) = 0$ (constant rule) (*b*) $dy/dx = 0$

(*c*) $y' = 7$ (linear function rule) (*d*) $D_x(25 - 6x) = -6$

5.4 Differentiate each of the following functions, using the power function rule. Co the different notations.

(*a*) $y = 9x^4$ (*b*) $f(x) = -5x^7$ (*c*) $f(x) = 4x^{-3}$

(*d*) $y = -7x^{-4}$ (*e*) $y = \frac{5}{x} = 5x^{-1}$ (*f*) $f(x) = \frac{1}{8x^3} =$

(*g*) $f(x) = 25\sqrt{x} = 25x^{1/2}$ (*h*) $y = \frac{1}{\sqrt{x^3}} = x^{-3/2}$

(*a*) $$\frac{d}{dx}(9x^4) = 36x^3$$

(*b*) $$f' = -35x^6$$

(*c*) $$f'(x) = 4(-3) \cdot x^{[-3-(1)]} = -12x^{-4} = \frac{-12}{x^4}$$

(*d*) $$\frac{dy}{dx} = -7(-4) \cdot x^{[-4-(1)]} = \frac{28}{x^5}$$

(*e*) $$D_x(5x^{-1}) = 5(-1x^{-2}) = -5x^{-2} = \frac{-5}{x^2}$$

(*f*) $$\frac{d}{dx}\left(\frac{1}{8}x^{-3}\right) = \frac{1}{8}(-3x^{-4}) = -\frac{3}{8}x^{-4} = \frac{-3}{8x^4}$$

(*g*) $$\frac{df}{dx} = 25 \cdot \tfrac{1}{2}x^{[(1/2)-1]} = 12.5x^{-1/2} = \frac{12.5}{\sqrt{x}}$$

(*h*) $$\frac{d}{dx}(x^{-3/2}) = -\tfrac{3}{2}x^{[-(3/2)-1]} = -\tfrac{3}{2}x^{-5/2} = -1.5x^{-2.5} = \frac{-1.5}{\sqrt{x^5}}$$

5.5 Use the rule for sums and differences to differentiate the following functions, treating the dependent variable on the left as you would y and the independent variable on the right as you would x.

(a) $R = 6t^2 + 11t - 9$ (b) $C = 5t^3 - 8t^2 + 13t - 45$ (c) $p = 7q^5 - 9q^3$

(d) $q = 6p^4 + 13p^{-3}$ (e) $u = \dfrac{1}{x^2} - \dfrac{6}{x^3} = x^{-2} - 6x^{-3}$

(f) $v = \dfrac{1}{t} + \dfrac{1}{9t^4} = t^{-1} + \dfrac{1}{9}t^{-4}$ (g) $u = \sqrt[3]{t} + \dfrac{7}{\sqrt{t}} = t^{1/3} + 7t^{-1/2}$

(a) $$\frac{dR}{dt} = 12t + 11$$

(b) $$C' = 15t^2 - 16t + 13$$

(c) $$\frac{dp}{dq} = 35q^4 - 27q^2$$

(d) $$D_p(6p^4 + 13p^{-3}) = 24p^3 - 39p^{-4} = 24p^3 - \frac{39}{p^4}$$

(e) $$\frac{d}{dx}(x^{-2} - 6x^{-3}) = -2x^{-3} - 6(-3x^{-4})$$
$$= -2x^{-3} + 18x^{-4} = \frac{-2}{x^3} + \frac{18}{x^4}$$

(f) $$D_t\left(t^{-1} + \frac{1}{9}t^{-4}\right) = -t^{-2} - \frac{4}{9}t^{-5} = \frac{-1}{t^2} - \frac{4}{9t^5}$$

(g) $$\frac{d}{dt}(t^{1/3} + 7t^{-1/2}) = \frac{1}{3}t^{-2/3} - \frac{7}{2}t^{-3/2} = \frac{1}{3\sqrt[3]{t^2}} - \frac{7}{2\sqrt{t^3}}$$

THE PRODUCT RULE

5.6 Given $y = f(x) = 6x^3(5x + 4)$, (a) find the derivative directly, using the product rule; (b) simplify the original function first and then find the derivative; (c) compare the two derivatives.

(a) Recalling the formula for the product rule from (5.1),

$$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

let $g(x) = 6x^3$ and $h(x) = 5x + 4$. Then $g'(x) = 18x^2$ and $h'(x) = 5$. Substituting these values in the product rule formula,

$$y' = f'(x) = 6x^3(5) + (5x + 4)(18x^2)$$

and then simplifying algebraically, remembering from Section 1.3 to *add* exponents in multiplication,

$$y' = 30x^3 + 90x^3 + 72x^2 = 120x^3 + 72x^2$$

(b) First simplifying the original function by multiplication,

$$y = 6x^3(5x + 4) = 30x^4 + 24x^3$$

Then taking the derivative of the simplified function,

$$y' = 120x^3 + 72x^2$$

(c) The derivatives are the same, indicating that the derivative of a product can be found by either method. With more complex functions, however, the product rule is all but essential. Note that knowledge of a second method can be used to check answers.

5.7 Redo Problem 5.6, given $y = f(x) = (x^6 + 10)(x^5 - 8)$.

(a) Let $g(x) = x^6 + 10$ and $h(x) = x^5 - 8$. Then $g'(x) = 6x^5$ and $h'(x) = 5x^4$. Substit (5.1),

$$y' = f'(x) = (x^6 + 10)(5x^4) + (x^5 - 8)(6x^5)$$
$$= 5x^{10} + 50x^4 + 6x^{10} - 48x^5 = 11x^{10} - 48x^5 + 50x^4$$

(b) Simplifying first through multiplication,

$$y = (x^6 + 10)(x^5 - 8) = x^{11} - 8x^6 + 10x^5 - 80$$

Then taking the derivative,

$$y' = 11x^{10} - 48x^5 + 50x^4$$

(c) The answers will always be the same if done correctly.

5.8 Differentiate each of the following functions, using the product rule. *Note* problems has deliberately been kept simple in this and other sections to enab see how various procedures work. While proper and often easier to sim algebraically before taking the derivative, applying the rules to the problems the long run help students to master techniques more efficiently.

(a) $y = (6x^2 - 7)(3x^4)$ (b) $y = (5x^3 + 8x^2)(4x^5 - 2)$

(c) $y = (2x^3 - 4x^2 + 7x)(x^2 - 9)$ (d) $y = 10\sqrt{x}(6x^2 - 7)$

(a)
$$y' = (6x^2 - 7)(12x^3) + (3x^4)(12x)$$
$$= 72x^5 - 84x^3 + 36x^5 = 108x^5 - 84x^3$$

(b)
$$y' = (5x^3 + 8x^2)(20x^4) + (4x^5 - 2)(15x^2 + 16x)$$
$$= 100x^7 + 160x^6 + 60x^7 + 64x^6 - 30x^2 - 32x$$
$$= 160x^7 + 224x^6 - 30x^2 - 32x$$

(c)
$$y' = (2x^3 - 4x^2 + 7x)(2x) + (x^2 - 9)(6x^2 - 8x + 7)$$
$$= 4x^4 - 8x^3 + 14x^2 + 6x^4 - 8x^3 + 7x^2 - 54x^2 + 72x - 63$$
$$= 10x^4 - 16x^3 - 33x^2 + 72x - 63$$

(d) First convert the radical to a power function,

$$y = 10x^{1/2}(6x^2 - 7)$$

and then take the derivative,

$$y' = (10x^{1/2})(12x) + (6x^2 - 7)(5x^{-1/2})$$
$$= 120x^{3/2} + 30x^{3/2} - 35x^{-1/2} = 150x^{3/2} - 35x^{-1/2} = 150\sqrt{x^3} - \frac{35}{\sqrt{5}}$$

THE QUOTIENT RULE

5.9 Given

$$y = \frac{24x^6 + 18x^5}{3x^2}$$

(a) find the derivative directly, using the quotient rule; (b) simplify the function f take the derivative; (c) compare the two derivatives.

(a) From (5.2), the quotient rule formula is

$$f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

where $g(x)$ = the numerator $= 24x^6 + 18x^5$ and $h(x)$ = the denominator $= 3x^2$. Take tl derivatives,

$$g'(x) = 144x^5 + 90x^4 \qquad h'(x) = 6x$$

Then substitute these values in the quotient rule formula,

$$y' = \frac{3x^2(144x^5 + 90x^4) - (24x^6 + 18x^5)(6x)}{(3x^2)^2}$$

$$= \frac{432x^7 + 270x^6 - 144x^7 - 108x^6}{9x^4} = \frac{288x^7 + 162x^6}{9x^4}$$

$$= 32x^3 + 18x^2$$

(*b*) Simplify the original function first by division,

$$y = \frac{24x^6 + 18x^5}{3x^2} = 8x^4 + 6x^3$$

$$y' = 32x^3 + 18x^2$$

(*c*) The derivatives will always be the same if done correctly, but as functions grow in complexity the quotient rule becomes more important. Remember, too, that an alternate method always remains a way of checking answers.

5.10 Differentiate each of the following functions, using the quotient rule. Continue to apply the rules to the functions as given. Later, when all the rules have been mastered, the functions can be simplified first and the easiest rule applied.

(*a*) $y = \dfrac{7x^3}{4x + 9}$ (*b*) $y = \dfrac{10x^4}{x^2 + 8x + 25}$ (*c*) $y = \dfrac{4x^3 - 11}{3x^2 + 7}$

(*d*) $y = \dfrac{2x^3 - 4x^2 + 3}{5x^3 + 6}$

(*a*) Here $g(x) = 7x^3$, $h(x) = 4x + 9$, $g'(x) = 21x^2$, and $h'(x) = 4$. Substitute these values in the quotient rule formula,

$$y' = \frac{(4x + 9)(21x^2) - (7x^3)(4)}{(4x + 9)^2}$$

$$= \frac{84x^3 + 189x^2 - 28x^3}{(4x + 9)^2} = \frac{56x^3 + 189x^2}{(4x + 9)^2}$$

(*b*)

$$\frac{dy}{dx} = \frac{(x^2 + 8x + 25)(40x^3) - 10x^4(2x + 8)}{(x^2 + 8x + 25)^2}$$

$$= \frac{40x^5 + 320x^4 + 1000x^3 - 20x^5 - 80x^4}{(x^2 + 8x + 25)^2}$$

$$= \frac{20x^5 + 240x^4 + 1000x^3}{(x^2 + 8x + 25)^2}$$

(*c*)

$$\frac{dy}{dx} = \frac{(3x^2 + 7)(12x^2) - (4x^3 - 11)(6x)}{(3x^2 + 7)^2}$$

$$= \frac{36x^4 + 84x^2 - 24x^4 + 66x}{(3x^2 + 7)^2} = \frac{12x^4 + 84x^2 + 66x}{(3x^2 + 7)^2}$$

(*d*)

$$y' = \frac{(5x^3 + 6)(6x^2 - 8x) - (2x^3 - 4x^2 + 3)(15x^2)}{(5x^3 + 6)^2}$$

$$= \frac{30x^5 - 40x^4 + 36x^2 - 48x - 30x^5 + 60x^4 - 45x^2}{(5x^3 + 6)^2}$$

$$= \frac{20x^4 - 9x^2 - 48x}{(5x^3 + 6)^2}$$

THE GENERALIZED POWER FUNCTION RULE

5.11 Given $y = (4x + 9)^2$, (*a*) find the derivative directly, using the generalized po (*b*) Simplify the function first by squaring it and then take the derivative. results.

(*a*) From the generalized power function rule in (5.3), given $f(x) = [g(x)]^n$,

$$f'(x) = n[g(x)]^{n-1} \cdot g'(x)$$

Here $g(x) = 4x + 9$, $g'(x) = 4$, and $n = 2$. Substitute these values in the generaliz rule,

$$y' = 2(4x + 9)^{2-1} \cdot 4 = 8(4x + 9) = 32x + 72$$

(*b*) Square the function first and then take the derivative,

$$y = (4x + 9)(4x + 9) = 16x^2 + 72x + 81$$
$$y' = 32x + 72$$

(*c*) The derivatives are identical but the generalized power function rule is faster and n higher, negative, and fractional values of n.

5.12 Find the derivative for each of the following functions, using the generalized p rule:

(*a*) $y = (2x^3 + 7)^5$ (*b*) $y = (x^2 - 7x + 4)^3$ (*c*) $y = \dfrac{1}{4x^3 + 8x + 5}$

(*d*) $y = \sqrt{16 - x^2}$ (*e*) $y = \dfrac{1}{\sqrt{3x^2 + 7}}$

(*a*) Here $g(x) = 2x^3 + 7$, $g'(x) = 6x^2$, and $n = 5$. Substitute these values in the gene function rule,

$$y' = 5(2x^3 + 7)^{5-1} \cdot 6x^2$$
$$= 5(2x^3 + 7)^4 \cdot 6x^2 = 30x^2(2x^3 + 7)^4$$

(*b*)

$$y' = 3(x^2 - 7x + 4)^2 \cdot (2x - 7) = 3(2x - 7)(x^2 - 7x + 4)^2$$

(*c*) First convert the function to an easier equivalent form,

$$y = (4x^3 + 8x + 5)^{-1}$$

then use the generalized power function rule,

$$y' = -1(4x^3 + 8x + 5)^{-2} \cdot (12x^2 + 8) = -(12x^2 + 8)(4x^3 + 8x + 5)^{-2}$$
$$= \frac{-(12x^2 + 8)}{(4x^3 + 8x + 5)^2}$$

(*d*) Convert the radical to a power function, then differentiate,

$$y = (16 - x^2)^{1/2}$$
$$y' = \tfrac{1}{2}(16 - x^2)^{-1/2} \cdot (-2x)$$
$$= -x(16 - x^2)^{-1/2} = \frac{-x}{\sqrt{16 - x^2}}$$

(*e*) Convert to an equivalent form; then take the derivative,

$$y = (3x^2 + 7)^{-1/2}$$
$$y' = -\tfrac{1}{2}(3x^2 + 7)^{-3/2} \cdot (6x)$$
$$= -3x(3x^2 + 7)^{-3/2} = \frac{-3x}{(3x^2 + 7)^{3/2}} = \frac{-3x}{\sqrt{(3x^2 + 7)^3}}$$

THE CHAIN RULE

5.13 For practice, use the chain rule expressed in (*5.4*) to find the derivative for each of the following functions. Check the answers on your own, using the generalized power function rule.

(*a*) $f(x) = (12x + 9)^4$ (*b*) $f(x) = 2(4x^2 + 9)^3$ (*c*) $f(x) = 5(3x^2 + 7x + 9)^4$

(*a*) From (*5.4*), if $f(x) = g[h(x)]$, $f'(x) = g'[h(x)] \cdot h'(x)$. Let

$$g(x) = x^4 \quad \text{and} \quad h(x) = 12x + 9$$

then

$$g'(x) = 4x^3 \quad \text{and} \quad h'(x) = 12$$

Substitute $h(x)$ for x in $g'(x)$ prior to using the chain rule,

$$g'[h(x)] = 4(12x + 9)^3$$

then substitute these values in the chain rule formula (*5.4*),

$$f'(x) = 4(12x + 9)^3 \cdot 12 = 48(12x + 9)^3$$

(*b*) Let

$$g(x) = 2x^3 \quad \text{and} \quad h(x) = 4x^2 + 9$$

then

$$g'(x) = 6x^2 \quad \text{and} \quad h'(x) = 8x$$

First substitute $h(x)$ for x in $g'(x)$,

$$g'[h(x)] = 6(4x^2 + 9)^2$$

then substitute the proper values in the chain rule formula,

$$f'(x) = 6(4x^2 + 9)^2 \cdot (8x) = 48x(4x^2 + 9)^2$$

(*c*) Let $g(x) = 5x^4$ and $h(x) = 3x^2 + 7x + 9$, then $g'(x) = 20x^3$, $h'(x) = 6x + 7$, and $g'[h(x)] = 20(3x^2 + 7x + 9)^3$. Substituting,

$$f'(x) = 20(3x^2 + 7x + 9)^3 \cdot (6x + 7)$$
$$= (120x + 140)(3x^2 + 7x + 9)^3$$

5.14 For further practice, use the chain rule given in (*5.5*) to differentiate the following functions. Check the answers on your own, again by using the generalized power function rule.

(*a*) $y = (13x - 4)^6$ (*b*) $y = 3(5x^2 + 11)^4$ (*c*) $y = 10(3x^3 + 13)^{-4}$

(*a*) From (*5.5*), if $y = f(u)$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Let $y = u^6$ and $u = 13x - 4$, then

$$\frac{dy}{du} = 6u^5 \quad \text{and} \quad \frac{du}{dx} = 13$$

Substitute in the second chain rule formula from (*5.5*),

$$\frac{dy}{dx} = 6u^5 \cdot 13 = 78u^5$$

then replace u with $13x - 4$ to have the answer exclusively in terms of x,

$$\frac{dy}{dx} = 78(13x - 4)^5$$

(*b*) Let

$$y = 3u^4 \quad \text{and} \quad u = 5x^2 + 11$$

then

$$\frac{dy}{du} = 12u^3 \quad \text{and} \quad \frac{du}{dx} = 10x$$

Substituting in the formula,

$$\frac{dy}{dx} = 12u^3 \cdot 10x = 120xu^3$$

then replacing u with $5x^2 + 11$,

$$\frac{dy}{dx} = 120x(5x^2 + 11)^3$$

(c) Let $y = 10u^{-4}$ and $u = 3x^3 + 13$

then $\frac{dy}{du} = -40u^{-5}$ and $\frac{du}{dx} = 9x^2$

Substituting, $\frac{dy}{dx} = -40u^{-5} \cdot 9x^2 = -360x^2u^{-5} = -360x^2(3x^3 + 13)^{-5}$

COMBINATION OF RULES

5.15 Use whatever combination of rules is necessary to find the derivatives (functions. Again, do not simplify the original functions. They are deliberatel ease the algebraic computation.

(a) $y = (6x + 8)(4x + 9)^5$ (b) $y = \frac{4x(3x - 5)}{2x + 1}$ (c) $y = (4x - 5$

(d) $y = \frac{(4x - 5)^3}{6x + 7}$ (e) $y = \left(\frac{5x - 2}{3x + 4}\right)^3$

(a) The function involves a product in which one function is raised to a power. Hence rule and the generalized power function rule are needed. Starting with the prod

$$y' = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

where $g(x) = 6x + 8$ $h(x) = (4x + 9)^5$ and $g'(x) = 6$

Using the generalized power function rule for $h'(x)$,

$$h'(x) = 5(4x + 9)^4 \cdot 4 = 20(4x + 9)^4$$

Substitute the appropriate values in the product rule,

$$y' = (6x + 8) \cdot 20(4x + 9)^4 + (4x + 9)^5 \cdot (6)$$

and simplify algebraically,

$$y' = 20(6x + 8)(4x + 9)^4 + 6(4x + 9)^5$$
$$= (120x + 160)(4x + 9)^4 + 6(4x + 9)^5$$

(b) The function involves a quotient with a product in the numerator. Hence both th and the product rule are required. Starting with the quotient rule,

$$y' = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

where $g(x) = 4x(3x - 5)$ $h(x) = 2x + 1$ $h'(x) = 2$

and using the product rule for $g'(x)$,

$$g'(x) = 4x \cdot 3 + (3x - 5) \cdot 4$$
$$= 12x + 12x - 20 = 24x - 20$$

substitute the appropriate values in the quotient rule,

$$y' = \frac{(2x + 1)(24x - 20) - [4x(3x - 5)] \cdot 2}{(2x + 1)^2}$$

Simplify algebraically,

$$y' = \frac{48x^2 - 40x + 24x - 20 - 24x^2 + 40x}{(2x+1)^2} = \frac{24x^2 + 24x - 20}{(2x+1)^2}$$

Note: To check this answer one could let

$$y = 4x \cdot \frac{3x-5}{2x+1} \qquad \text{or} \qquad y = \frac{4x}{2x+1} \cdot (3x-5)$$

and use the product rule involving a quotient.

(*c*) Here we have a product involving a quotient. The product and quotient rules will both be needed, therefore. Starting with the product rule,

$$y' = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

where $\qquad g(x) = 4x - 5 \qquad h(x) = \dfrac{2x^5}{3x+2} \qquad g'(x) = 4$

and using the quotient rule for $h'(x)$,

$$h'(x) = \frac{(3x+2)(10x^4) - (2x^5) \cdot (3)}{(3x+2)^2}$$

$$= \frac{30x^5 + 20x^4 - 6x^5}{(3x+2)^2} = \frac{24x^5 + 20x^4}{(3x+2)^2}$$

substitute the appropriate values in the product rule,

$$y' = (4x-5) \cdot \frac{24x^5 + 20x^4}{(3x+2)^2} + \frac{2x^5}{3x+2} \cdot (4)$$

$$= \frac{96x^6 - 40x^5 - 100x^4}{(3x+2)^2} + \frac{8x^5}{3x+2} = \frac{120x^6 - 24x^5 - 100x^4}{(3x+2)^2}$$

Note: To check this answer, one could let $y = [2x^5(4x-5)]/(3x+2)$, and use the quotient rule involving a product.

(*d*) Starting with the quotient rule, where $g(x) = (4x-5)^3$, $h(x) = 6x + 7$, $h'(x) = 6$, and, using the generalized power function rule for $g'(x)$, $g'(x) = 3(4x-5)^2 \cdot 4 = 12(4x-5)^2$. Substitute these values in the quotient rule,

$$y' = \frac{(6x+7) \cdot 12(4x-5)^2 - (4x-5)^3 \cdot 6}{(6x+7)^2}$$

$$= \frac{(72x+84)(4x-5)^2 - 6(4x-5)^3}{(6x+7)^2}$$

To check this answer, one could let $y = (4x-5)^3 \cdot (6x+7)^{-1}$, and use the product rule involving the generalized power function rule twice.

(*e*) Start with the generalized power function rule,

$$y' = 3\left(\frac{5x-2}{3x+4}\right)^2 \cdot \frac{d}{dx}\left(\frac{5x-2}{3x+4}\right) \tag{5.8}$$

Then, using the quotient rule,

$$\frac{d}{dx}\left(\frac{5x-2}{3x+4}\right) = \frac{(3x+4)(5) - (5x-2)(3)}{(3x+4)^2} = \frac{26}{(3x+4)^2}$$

substitute this value in (*5.8*),

$$y' = 3\left(\frac{5x-2}{3x+4}\right)^2 \cdot \frac{26}{(3x+4)^2} = \frac{78(5x-2)^2}{(3x+4)^4}$$

To check this answer, let $y = (5x-2)^3 \cdot (3x+4)^{-3}$, and use the product rule involving the generalized power function rule twice.