550.681 Handout7
Patrick Bindjeme

Theorem on Truncated Power Functions

Every member of $N_n^{2m+1}$ has a representation
$S(x) = \sum_{i=0}^{m} a_i x^i + \sum_{i=0}^{n} b_i (x-t_i)^{2m+1}$ in which $\sum_{i=0}^{n} b_i t_i^j = 0$ for $0 \leq j \leq m$.

Examples

1. Determine the natural cubic spline that interpolates the table

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

$S(x) = a_0 + a_1 x + b_0 x_0^3 + b_1 (x-1)_+^3 + b_2 (x-2)_+^3 + b_3 (x-3)_+^3$, with

- $b_0 + b_1 + b_2 + b_3 = 0$
- $b_1 + 2b_2 + 3b_3 = 0$
- $S(0) = a_0 = 1$
- $S(1) = a_0 + a_1 + b_0 = 1$
- $S(2) = a_0 + 2a_1 + 8b_0 + b_1 = 0$
- $S(3) = a_0 + 3a_1 + 27b_0 + 8b_1 + b_2 = 10$

so $a_0 = 1, a_1 = 15/14, b_0 = -(15/14), b_1 = 38/7, b_2 = -(107/14), b_3 = 23/7$, so

$S(x) = 1 + \frac{15}{14} x - \frac{15}{14} x_+^3 + \frac{38}{7} (x-1)_+^3 - \frac{107}{14} (x-2)_+^3 + \frac{23}{7} (x-3)_+^3$

Theorem on Uniqueness of Natural Splines of Odd Degree

Let $t_0 < t_1 < \ldots < t_n$ be given, and let $0 \leq m \leq n$. There is a unique natural spline of degree $2m + 1$ taking prescribed values at the knots.

$p$ be the polynomial of degree at most $n$ that interpolates a function $f$ at a set of $n+1$ distinct nodes, $x_0, x_1, \ldots, x_n$. Then, if $t \notin \{x_0, x_1, \ldots, x_n\}$, $f(t) - p(t) = f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t-x_j)$.

Theorem on Derivatives and Divided Differences

If $f$ is $n$ times continuously differentiable on $[a, b]$ and if $x_0, x_1, \ldots, x_n$ are distinct points in $[a, b]$, then there exists a point $\xi$ in $(a, b)$ such that

$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$.

Newton Divided Difference Method

$p(1) = 2, p'(1) = 3, \text{ and } p(2) = 6, p'(2) = 7, p''(2) = 8.$

$f[1, 1] = 3, f[1, 2] = \frac{2-6}{1-2} = 4, f[2, 2] = 7, f[1, 1, 2] = \frac{3-4}{1-2} = 1, f[1, 2, 2] = \frac{4-7}{2-2} = 3, f[2, 2, 2] = \frac{1}{2}(8) = 4, f[1, 1, 2, 2] = \frac{3-2}{1-2} = 2, f[1, 2, 2, 2] = \frac{4-3}{2-2} = 1, f[1, 1, 2, 2, 2] = \frac{5-4}{2-2} = -1.$
Natural Cubic Spline

\[
S(x) = \begin{cases} 
S_0(x) & x \in [t_0, t_1] \\
S_1(x) & x \in [t_1, t_2] \\
\vdots & \\
S_{n-1}(x) & x \in [t_{n-1}, t_n]
\end{cases},
\]

where \( S \) is assumed to be twice continuously differentiable.

\[
S_i(x) = \frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 + \left( \frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6} \right) (x - t_i) + \left( \frac{y_i}{h_i} - \frac{z_ih_i}{6} \right) (t_{i+1} - x),
\]

with \( h_i = t_{i+1} - t_i, \ z_0 = z_n = 0, \) and \( h_{i-1}z_{i-1} + 2(h_i + h_{i-1})z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1}). \)

1. \[
\begin{array}{cccc}
x & -1 & 0 & 1 \\
y & 5 & 7 & 9 \\
\end{array}
\]

\( h_0 = h_1 = 1, \ 4z_1 = 6(9 - 7) - 6(7 - 5) = 0, \) so \( z_0 = z_1 = z_2 = 0, \) and

\[
S(x) = \begin{cases} 
7(x + 1) - 5x & x \in [-1, 0] \\
9x + 7(1 - x) & x \in [0, 1]
\end{cases},
\]

so \( S(x) = 2x + 7. \)

2. \[
\begin{array}{cccc}
x & 0 & 1 & 2 & 3 \\
y & 1 & 4 & 10 & 0 \\
\end{array}
\]

\( h_0 = h_1 = h_2 = 1, \ \begin{cases} 
4z_1 + z_2 = 6(0 - 1) - 6(1 - 1) \\
z_1 + 4z_2 = 6(10 - 0) - 6(0 - 1)
\end{cases}, \) so \( z_1 = -6, \ z_2 = 18, \) and

\[
S(x) = \begin{cases} 
1 + x - x^3 & x \in [0, 1] \\
1 - 2(x - 1) - 3(x - 1)^2 + 4(x - 1)^3 & x \in [1, 2] \\
4(x - 2) + 9(x - 2)^2 - 3(x - 2)^3 & x \in [2, 3]
\end{cases}
\]