Problem 1
In the two methods we have studied in the class for constructing cubic splines, we need to impose two end-conditions to pin down a particular cubic spline. Set up two equations for the parameters \(d_1, d_2, \ldots, d_n\) in the first method from the following conditions:

(a) Hermite conditions: \(S(x_1) = f'(x_1)\) and \(S(x_n) = f'(x_n)\).

(b) Natural cubic spline conditions: \(S''(x_1) = S''(x_n) = 0\).

(c) Periodic conditions: \(S'(x_1) = S'(x_n)\) and \(S''(x_1) = S''(x_n)\). (Assume \(f(x_1) = f(x_n)\) implicitly.)

Problem 2
Do the same as in Problem 1 for parameters \(m_1, m_2, \ldots, m_n\) in the second method.

Problem 3
Any cubic spline with knots \(x_1 < x_2 < \cdots < x_n\) can be expressed as:

\[ S(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \alpha_2 (x - x_2)^3 + \cdots + \alpha_{n-1} (x - x_{n-1})^3. \]

Let \(f(x) = \frac{1}{1+x^2}\) and we want to do least-squares fitting to \(f\) at nodes \(y_k = \frac{k}{2}, k = 0, 1, \ldots, 6\) from the family of cubic splines with knots \(x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3\). Use the above expression to set up the linear least-squares problem. (note: You are required just to set up the problem. Solving the problem is optional.)

Problem 4
(a) Show that any \(k\)-degree spline with \(k \geq 1\) and knots \(x_1 < x_2 < \cdots < x_n\) can be expressed in the form:

\[ S(x) = S_1(x) + \alpha_2 (x - x_2)^k + \cdots + \alpha_{n-1} (x - x_{n-1})^k. \]

where \(S_1(x)\) is a polynomial of degree \(k\).

(b) Use the expression in (a) to design an algorithm to find a continuous piecewise linear function \(S\) such that \(S(x_i) = f_i\) for \(i = 1, 2, \ldots, n\).