Centerpoints: A link between optimization and convex geometry

Amitabh Basu[1] and Timm Oertel[2]

[1] Johns Hopkins University
[2] ETH Zürich

Abstract. We introduce a concept that generalizes several different notions of a "centerpoint" in the literature. We develop an oracle-based algorithm for convex mixed-integer optimization based on centerpoints. Further, we show that algorithms based on centerpoints are "best possible" in a certain sense. Motivated by this, we establish several structural results about this concept and provide efficient algorithms for computing these points.

1 Introduction

Let μ be a Borel-measure on $\mathbb{R}^n$ such that $0 < \mu(\mathbb{R}^n) < \infty$. Without any loss of generality, we normalize the measure to be a probability measure, i.e., $\mu(\mathbb{R}^n) = 1$. For $S \subset \mathbb{R}^n$ closed, we define the set of *centerpoints* $\mathcal{C}(S, \mu) \subset S$ as the set that attains the following maximum

$$\mathcal{F}(S, \mu) := \max_{x \in S} \inf_{u \in \mathcal{S}^{n-1}} \mu(H^+(u, x)), \tag{1}$$

where $\mathcal{S}^{n-1}$ denotes the $(n-1)$-dimensional sphere and $H^+(u, x)$ denotes the half-space $\left\{y \in \mathbb{R}^n \mid u^\mathsf{T}(y - x) \geq 0\right\}$. This definition unifies several definitions from convex geometry and statistics. Two notable examples are:

1. *Winternitz measure of symmetry.* Let μ be the Lebesgue measure restricted to a compact convex body K, or equivalently, the uniform probability measure on K, and let $S = \mathbb{R}^n$. $\mathcal{F}(S, \mu)$ in this setting is known in the literature as the *Winternitz measure of symmetry* of K, and the centerpoints $C(S, \mu)$ are the "points of symmetry" of K. This notion was studied by Grünbaum in [12] and surveyed by the same author (along with other measures of symmetry for convex bodies) in [13].
2. *Tukey depth and median.* In statistics and computational geometry, the function $f_\mu : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f_\mu(x) := \inf_{u \in \mathcal{S}^{n-1}} \mu(H^+(u, x)) \tag{2}$$

is known as the *halfspace depth function* or the *Tukey depth function* for the measure μ, first introduced by Tukey [23]. Taking $S = \mathbb{R}^n$, the centerpoints

$C(\mathbb{R}^n, \mu)$ are known as the *Tukey medians* of the probability measure μ, and $\mathcal{F}(\mathbb{R}^n, \mu)$ is known as the maximum *Tukey depth* of μ. Tukey introduced the concept when μ is a finite sum of Dirac measures (i.e., a finite set of points with the counting measure), but the concept has been generalized to other probability measures and analyzed from structural, as well as computational perspectives. See [18] for a survey of structural aspects and other notions of "depth" used in statistics, and [9] and the references therein for a survey and recent approaches to computational aspects of the Tukey depth.

Our Results. To the best of our knowledge, all related notions of centerpoints in the literature always insist on the set S being $\mathbb{R}^n$, i.e., the centerpoint can be any point from the Euclidean space. We consider more general S, as this captures certain operations performed by oracle based mixed-integer convex optimization algorithms. In Section 2, we elaborate on this connection between centerpoints and algorithms for mixed-integer optimization. We first give an algorithm for solving convex mixed-integer optimization given access to first-order (separation) oracles, based on centerpoints. Second, we show that oracle-based algorithms for convex mixed-integer optimization that use centerpoint information are "best possible" in a certain sense. This comprises our main motivation in studying centerpoints.

In Section 4, we show that when $S = \mathbb{R}^n$ and μ is the uniform measure on polytopes, centerpoints are unique, a question which was surprisingly not proved earlier. We also present a new technique to lower bound $\mathcal{F}(\mathbb{Z}^n \times \mathbb{R}^d, \nu)$ where ν is the "mixed-integer" uniform measure on $K \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ and K is some compact, convex set. Such bounds immediately imply bounds on the complexity of oracle-based convex mixed-integer optimization algorithms.

In Section 5, we present a number of exact and approximation algorithms for computing centerpopints. To the best of our knowledge, the computational study of centerpoints has only been done for measures μ that are a finite sum of Dirac measures, i.e., for finite point sets, or when μ is the uniform measure on two dimensional polygons (e.g. see [6] and the references therein). We initiate a study for other measures; in particular, the uniform measure on a compact convex body, the counting measure on the integer points in a compact convex body, and the mixed-integer volume of the mixed-integer points in a compact convex body. All our algorithms are exponential in the dimension n but polynomial in the remaining input data, so these are polynomial time algorithms if n is assumed to be a constant. Algorithms that are polynomial in n are likely to not exist because of the reduction to the so-called *closed hemisphere problem* – see Chapter 7 in Bremner, Fukuda and Rosta [17].

2 An application to mixed-integer optimization

We will be interested in the setting when the measure μ is based on the mixed-integer volume of the mixed-integer points in a compact convex body K, and S is the set of mixed-integer points in K. More precisely, let $K \subseteq \mathbb{R}^n \times \mathbb{R}^d$ be a convex set. Let vol_d be the d-dimensional volume (Lebesgue measure). We define the *mixed-integer volume with respect to* K as

$$\mu_{mixed,K}(C) := \frac{\sum_{z\in\mathbb{Z}^n} \text{vol}_d(C \cap K \cap (\{z\} \times \mathbb{R}^d))}{\sum_{z\in\mathbb{Z}^n} \text{vol}_d(K \cap (\{z\} \times \mathbb{R}^d))} \tag{3}$$

for any Lebesgue measureable subset $C \subseteq \mathbb{R}^n \times \mathbb{R}^d$. For later use we want to introduce the notation $\bar{\mu}_{mixed}(C) = \sum_{z\in\mathbb{Z}^n} \text{vol}_d(C\cap(\{z\}\times\mathbb{R}^d))$. The dimensions n and d will be clear from the context.

Our main motivation to study centerpoints comes from its natural connection to convex mixed-integer optimization. Consider the following unconstrained optimization problem

$$\min_{(x,y)\in\mathbb{Z}^n\times\mathbb{R}^d} g(x, y). \tag{4}$$

where $g : \mathbb{R}^n \times \mathbb{R}^d \mapsto \mathbb{R}$ is a convex function given by a first-order evaluation oracle. Queried at a point the oracle return the corresponding function value and an element from the subdifferential. We assume that the problem is bounded. Further, if $d \neq 0$, we assume that for every fixed $x \in \mathbb{Z}^n$, $g(x, y)$ is Lipschitz continuous in the y variables with Lipschitz constant L. We present a general cutting plane method based on centerpoints, i.e. the *centerpoint-method*. This can be interpreted as an extension of the well-known Method of Centers of Gravity or other cutting plane methods such as the Ellipsoid method or Kelly's cutting plane method (see [19, Section 3.2.6.]) for convex optimization. This type of idea was also explored by Bertsimas and Vempala in [5] for continuous convex optimization.

We assume that we have access to (approximate) centerpoints of polytopes through an oracle. As in statistics, we introduce the notation

$$D_\mu(\alpha) := \{x \in \mathbb{R}^n : f_\mu(x) \geq \alpha\}. \tag{5}$$

We define the oracle for the case that we only have access to approximate centerpoints as follows.

Definition 1 (α-central-point-oracle). *For a polytope P, the oracle returns a point $z \in D_\mu(\alpha)$, where $\mu := \mu_{mixed,P}$.*

This way we hide the complexity of computing centerpoints in the oracle and keep the following discussion as general as possible. However, for several special cases the oracle can be realized as we discuss in subsequent sections.

The general algorithmic framework is as follows. We start with a bounding box, say $P^0 := [0, B]^{n+d}$ with $B \in \mathbb{Z}_+$, that is guaranteed to contain an optimum and initialize $x^\star = 0, y^\star = 0$. Then, we construct iteratively a sequence of polytopes $P^1, P^2, \ldots$ by intersecting P^k with the half-space defined by its (approximate) centerpoint and the corresponding subgradient arising from g. That is, let $(x_k, y_k) \in D_\mu(\alpha)(P_k)$ and let $h_k \in \partial g(x_k, y_k)$. Then we define $(x^\star, y^\star) := \text{argmin}\{g(x^\star, y^\star), g(x_k, y_k)\}$ and $P^{k+1} := \{(x, y) \in P^0 \mid g(x^\star, y^\star) - g(x_i, y_i) \geq h_i^\intercal(x - x_i, y - y_i),\ i = 1, \ldots, k\}$. It follows that

$$P^k \supset P^{k+1} \supset \underset{x\in\mathbb{Z}^n\times\mathbb{R}^d}{\text{argmin}}\ g(x)$$

for all $k \in \mathbb{N}$. Further, by the choice of (x_k, y_k), the measure of P^k decreases in each iteration by a fraction of at least $1-\alpha$. With $(x^\star, y^\star)$ we keep track of the (approximate) centerpoint x_k that has the smallest objective value $g^\star := g(x^\star, y^\star)$ among all points we encounter.

Let $(\hat{x}, \hat{y}) \in \mathbb{Z}^n \times \mathbb{R}^d$ attain the optimal value $\hat{g}$ of Problem (4). We have $\bar{\mu}_{mixed}(\{0,\dots,B\}^n \times [0,B]^d) \approx B^{n+d}$. By standard arguments, we can bound $\bar{\mu}_{mixed}(P_k)$ from below as follows

$$\begin{aligned}\bar{\mu}_{mixed}(P_k) &\geq \bar{\mu}_{mixed}(\{(x,y) \in \mathbb{Z}^n \times \mathbb{R}^d \mid g((x,y)) - \hat{g} \leq g^\star - \hat{g}\}) \\ &\geq \bar{\mu}_{mixed}\left(\left\{(\hat{x}, y) \in \{\hat{x}\} \times \mathbb{R}^d \;\middle|\; \|(\hat{x},\hat{y}) - (\hat{x},y)\|_2 \leq \frac{g^\star - \hat{g}}{L}\right\}\right) \\ &= \left(\frac{\hat{g} - g^\star}{L}\right)^d \kappa_d,\end{aligned}$$

where κ_d denotes the volume of the d-dimensional unit-ball. Then, it follows that after at most

$$k \leq \frac{d \ln\left(\frac{\epsilon}{LB}\right) + n \ln(B)}{\ln(1-\alpha)}$$

iterations we have that $g(x^\star, y^\star) - g(\hat{x}, \hat{y}) \leq \epsilon$. Note that in the pure integer case when $d = 0$ we can actually solve the problem exactly.

It is not difficult to generalize this to the constrained optimization case:

$$\min_{\substack{x \in \mathbb{Z}^n \times \mathbb{R}^d, \\ h(x) \leq 0}} g(x).$$

where $g, h : \mathbb{R}^n \times \mathbb{R}^d \mapsto \mathbb{R}$ are convex functions given by first-order oracles. Further, the algorithm can be extended to handle quasi-convex functions, if one has access to separation oracles for their sublevel sets.

The main feature of this approach is that, from the point view of the number of function oracle calls, this algorithm is best possible. Assume that $d = 0$ and that we can compute exact centerpoints. Then one can prove the following theorem.

Theorem 1. *No algorithm can exist for solving general bounded convex integer optimization problems, that needs fewer function oracle calls than the exact centerpoint-method in the worst case.*

We omit the proof from this extended abstract.

3 General Properties

In this section we first establish some analytic properties of f_μ. This will justify the use of "maximum" in (1), instead of a supremum. The main result of this section is a bound on the quality of the centerpoints based on Helly numbers. We will denote the complement of a set X by X^c. We begin with a useful lemma whose proof appears in the Appendix (Section 6.1).

Lemma 1. *For any probability measure μ, $f_\mu(x)$ defined in* (2) *is quasi-concave on $\mathbb{R}^n$ and upper semicontinuous. Moreover, given $\bar{x} \in \mathbb{R}^n$ and $\delta > 0$, let $\bar{u} \in \mathcal{S}^{n-1}$ be such that $\mu(H^+(\bar{u}, \bar{x})) \leq \inf_{u \in \mathcal{S}^{n-1}} \mu(H^+(u, \bar{x})) + \frac{\delta}{2}$. Then $\bar{u}$ strongly separates the set $\{x \in \mathbb{R}^n \mid f(x) \geq f(\bar{x}) + \delta\}$ and $\bar{x}$, i.e., $\bar{u} \cdot x < \bar{u} \cdot \bar{x}$ for all x such that $f(x) \geq f(\bar{x}) + \delta$.*

Remark 1. Lemma 1 shows that $\sup_{x \in S} f_\mu(x)$ is always attained. See Proposition 7 in [21] where this is discussed for $S = \mathbb{R}^n$. The generalization to any closed subset S is easy; see also Proposition 5 in [21] which states the for every $\alpha \geq 0$, the set $D_\mu(\alpha)$ given by (5) is compact.

Next we generalize a theorem well-known in the literature on half-space (Tukey) depth [21, Proposition 9]; this was earlier stated by Grünbaum [12, Theorem 1] for uniform probability measures on compact convex sets. In all of these works, the authors consider $S = \mathbb{R}^n$, as discussed in the introduction. Our generalization is to consider more general S. For this we define the Helly number of S. Let $\mathcal{K} := \{S \cap K \mid K \subset \mathbb{R}^n \text{ convex}\}$. Then the Helly-number $h = h(S) \in \mathbb{N}$ of S is defined as the smallest number such that the following property is satisfied for all finite subsets $\{C_1, \dots, C_m\} \subset \mathcal{K}$: If

$$C_{i_1} \cap \cdots \cap C_{i_h} \neq \emptyset \text{ for all } \{i_1, \dots, i_h\} \subset \{1, \dots, m\}$$

then

$$C_1 \cap \cdots \cap C_m \neq \emptyset.$$

If no such number exists, then $h(S) = \infty$. This extension of Helly's number was first considered by Hoffman [14], and has recently been studied in [1, 2, 7].

Theorem 2. *Let $S \subseteq \mathbb{R}^n$ be a closed subset and let μ be such that $\mu(\mathbb{R}^n \setminus S) = 0$. If $h(S) < \infty$, then $\mathcal{F}(S, \mu) \geq h(S)^{-1}$.*

The proof of this theorem appears in the Appendix (Section 6.2). By applying the well known bound for the mixed-integer Helly-number [14, 1, 7] we get the following Corollary.

Corollary 1. *$\mathcal{F}(\mathbb{Z}^n \times \mathbb{R}^d, \mu) \geq \frac{1}{2^n(d+1)}$ for any finite measure μ on $\mathbb{R}^{n+d}$ such that $\mu(\mathbb{R}^{n+d} \setminus (\mathbb{Z}^n \times \mathbb{R}^d)) = 0$. In particular, this holds for $\mu_{mixed,K}$ for any compact convex set $K \subseteq \mathbb{R}^n \times \mathbb{R}^d$.*

Remark 2. Let $K \subset \mathbb{R}^{n+d}$ be a compact convex body and let $\mu_{mixed,K}$ denote the mixed-integer volume with respect to K, as defined in (3). One can show that in this case the infimum in (1) and (2) is actually achieved.

4 Specialized Properties

For a general measure, the centerpoint may not be unique. We show however that when $S = \mathbb{R}^n$ and μ is the uniform measure on a polytope, the centerpoint is unique. Surprisingly, this question had not been investigated before, and as far as we know the question of uniqueness for the centerpoint for general compact convex bodies is open.

Proposition 1. *Let μ be the uniform measure on a full-dimensional polytope $P \subset \mathbb{R}^n$. Then $C(\mathbb{R}^n, \mu)$ is a singleton, i.e., the centerpoint is unique.*

The proof of Proposition 1 appears in the Appendix (Section 6.3).

Remark 3. With similar arguments one can show the proposition also holds for strictly convex sets. An interesting open question remains: Is the centerpoint of a rational polytope rational? If so, is the size of the centerpoint polynomially bounded in the size of an irredundant description of the rational polytope?

In the remaining section we want to improve the bound on $\mathcal{F}(\mathbb{Z}^n \times \mathbb{R}^d, \nu)$ coming from Helly numbers (Theorem 2 and Corollary 1) when ν is a mixed-integer measure. Better bounds have been obtained by Grünbraum by exploiting properties of the *centroid* of a convex body K, which is defined as $c_K := \int_K x \, \mathrm{d}x$, where the integral is taken with respect to the uniform measure μ on K. Grünbaum proved in [12] that $\mu(H^+(u, c_K)) \geq \left(\frac{n}{n+1}\right)^n \geq e^{-1}$ for any $u \in \mathcal{S}^{n-1}$, which immediately implies that $\mathcal{F}(\mathbb{R}^n, \mu) \geq e^{-1}$. This, of course, drastically improves the Helly bound of $\frac{1}{n+1}$. In the following we want to extend these improved bounds to the mixed-integer setting. Ideally, we would want to prove the following conjecture.

Conjecture 1. Let $S = \mathbb{Z}^n \times \mathbb{R}^d$ and let $\nu = \mu_{mixed,K}$ for some compact, convex set $K \in \mathbb{R}^{n+d}$. Then $\mathcal{F}(\mathbb{Z}^n \times \mathbb{R}^d, \nu) \geq \frac{1}{2^n} \left(\frac{d}{d+1}\right)^d$.

In a first step we consider convex sets K that have a large *lattice-width*, where the lattice-width is defined as $\omega(K) := \min_{z \in \mathbb{Z}^n \setminus \{0\}} \|z\|_{(K-K)^\star}$. As an auxiliary lemma, we show that for convex sets with large lattice width, the d-dimensional Lebesgue measure $\bar{\nu} := \bar{\mu}_{mixed}$ of $K \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ can be bounded by the $(d+n)$-dimensional Lebesgue measure $\bar{\mu}$ of K and vice versa. Note that in this case we do not normalize the measures. In the pure integer setting, i.e., $d = 0$, this connection is well known. However, to the best of our knowledge, this kind of result has never been proven for the mixed-integer setting nor explicitly with respect to the lattice width. The proof appears in the Appendix (Section 6.4). We denote the projection of a set $X \subset \mathbb{R}^{n+d}$ onto the first n coordinates by $X|_{\mathbb{R}^n}$.

Lemma 2. *Let $K \subset \mathbb{R}^{n+d}$ be a closed convex set with non-empty interior. Let ω denote the lattice-width of $K|_{\mathbb{R}^n}$. If $\omega > cn(n+d)^{7/2}\alpha n^n \sqrt{n}$ for a universal constant α and a $c \in \mathbb{N}$, then*

$$e^{-\frac{1}{c}} \leq \frac{\bar{\nu}(K \cap (\mathbb{Z}^n \times \mathbb{R}^d))}{\bar{\mu}(K)} \leq e^{\frac{1}{c}}.$$

For the following theorem we introduce the following technical rounding procedure. Let K be a full-dimensional convex body with a sufficiently large lattice width, i.e., $\omega(K) > cn(n+d)^{7/2}\alpha n^n \sqrt{n}$ for some positive integer c, where α is the constant from Lemma 2. Let μ be the uniform measure on K and let

$x^\star \in \mathcal{C}(\mathbb{R}^{n+d}, \mu)$. By Lemma 5, there exist $b_i \in (-x^\star + K) \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ for $i = 1, \ldots, n$ such that $b_1|_{\mathbb{R}^n}, \ldots, b_n|_{\mathbb{R}^n}$ is a Korkine-Zolotarev basis [15] of $\mathbb{Z}^n$ with respect to the maximum inscribed ellipsoid of $K|_{\mathbb{R}^n}$. In addition we define for $i = n+1, \ldots, n+d$ $b_i \in \mathbb{R}^{n+d}$ as the i-th unit vector. Hence, $b_1, \ldots, b_{n+d}$ define a basis of $\mathbb{R}^{n+d}$.

Given $x = \sum_{i=1}^{n+d} \lambda_i b_i \in \mathbb{R}^{n+d}$ with $\lambda_i \in \mathbb{R}$ for all i, we define $[x]_K \in \mathbb{Z}^n \times \mathbb{R}^d$ as $\sum_{i=1}^{n} \lfloor \lambda_i \rfloor b_i + \sum_{i=n+1}^{n+d} \lambda_i b_i$, i.e., we round x to a close mixed-integer point with respect to K.

Theorem 3. *Let $\nu := \mu_{mixed,K}$, where $K \subset \mathbb{R}^{n+d}$ is a compact convex body and $\nu(\mathbb{R}^{n+d}) \neq 0$, and let $x^\star$ be the centerpoint with respect to μ, the uniform measure on K. If $\omega > 2c(n+d)^{9/2}\alpha n^n\sqrt{n}$ for a universal constant α, then*

$$f_\nu([x^\star]_K) \geq e^{-1/c}(\mathcal{F}(\mathbb{R}^{d+n}, \mu) - e^{2/c} + 1).$$

Grünbaum's Theorem implies then, that $\mathcal{F}(\mathbb{Z}^n \times \mathbb{R}^d, \nu) \geq e^{-1/c-1} - e^{1/c} + e^{-1/c}$.

Proof. As before, let $\bar{\mu}$ denote the $(d+n)$-dimensional Lebesgue measure with respect to K and let $\bar{\nu}$ denote the d-dimensional Lebesgue measure with respect to $K \cap (\mathbb{Z}^n \times \mathbb{R}^d)$, i.e. they are not normalized.

In a first step we prove the following claim: $|\mu(H^+) - \nu(H^+)| \leq e^{2/c} - 1$ for any half-space H^+. This implies that $|\mathcal{F}(\mathbb{R}^{d+n}, \mu) - \mathcal{F}(\mathbb{R}^{d+n}, \nu)| \leq e^{2/c} - 1$, and, in particular, $|\mathcal{F}(\mathbb{R}^{d+n}, \mu) - f_\nu(x^\star)| \leq e^{2/c} - 1$.

Let H^+ be any half-space and let H^- denote its closed complement. The lattice-width of either $K \cap H^+$ or $K \cap H^-$ is larger or equal than $\omega/2$. Since both cases are similar, we only consider the case $\omega(K \cap H^-) \geq cn(n+d)\alpha n^n\sqrt{n}$. Then, by Lemma 2,

$$\begin{aligned}
\nu(H^+) = \frac{\bar{\nu}(K \cap H^+)}{\bar{\nu}(K)} &\leq \frac{e^{1/c}\bar{\mu}(K) - e^{-1/c}\bar{\mu}(K \cap H^-)}{e^{-1/c}\bar{\mu}(K)} \\
&= \frac{\bar{\mu}(K \cap H^+)}{\bar{\mu}(K)} + \frac{(e^{1/c} - e^{-1/c})\bar{\mu}(K)}{e^{-1/c}\bar{\mu}(K)} \\
&= \mu(H^+) + (e^{2/c} - 1).
\end{aligned}$$

Similarly we can derive a lower bound. This proves the claim.

In the second step we bound the error made by rounding the $x^\star$ to $[x^\star]_K$. By Lemma 5 and the choice of our rounding procedure, we know that $[x^\star]_K$ is contained in $\frac{1}{c(n+d)}(K - x^\star) + x^\star$. Hence, $[x^\star]_K + \frac{c(n+d)-1}{c(n+d)}(K - x^\star) \subset K \subset [x^\star]_K + \frac{c(n+d)+1}{c(n+d)}(K - x^\star)$. We have for any $u \in \mathcal{S}^{n+d-1}$ that $e^{-1/c}\nu(H^+(u, x^\star)) \leq \nu(H^+(u, [x^\star]_K)) \leq e^{1/c}\nu(H^+(u, x^\star))$. Together with the previous claim it follows that

$$f_\nu([x^\star]_K) \geq e^{-1/c}(\mathcal{F}(\mathbb{R}^{d+n}, \mu) - e^{2/c} + 1).$$

$\square$

5 Computational Aspects

All our algorithms all under the standard Turing machine model of computation. We say that $x \in S$ is an ϵ-centerpoint for S, μ, if $f_\mu(x) \geq \mathcal{F}(S, \mu) - \epsilon$ where $\mathcal{F}(S, \mu)$ is defined in (1) and f_μ is defined in (2).

5.1 Exact Algorithms

Uniform measure on polytopes. Since the rationality of the centerpoint for uniform measures on rational polytopes is an open question (see Remark 3), we consider an "exact" algorithm as one which returns an ϵ-centerpoint and runs in time polynomial in $\log(\frac{1}{\epsilon})$ and the size of the description of the rational polytope.

Theorem 4. *Let n be a fixed natural number. There is an algorithm which takes as input a rational polytope $P \subseteq \mathbb{R}^n$ and $\epsilon > 0$, and returns an ϵ-centerpoint for $\mathbb{R}^n, \mu$. The algorithms runs in time polynomial in the size of an irredundant description of P and $\log(\frac{1}{\epsilon})$.*

Proof. Since the f_μ defined in (2) is quasi-concave by Lemma 1, a x^* satisfying $f_\mu(x) \geq \mathcal{F}(S, \mu) - \epsilon$ if one has an approximate evaluation oracle for f_μ, and an approximate separation oracle for the level sets $D_\mu(\alpha)$ [10]. Moreover, the number of oracle calls made is bounded by a polynomial in the size of an irredundant description of P and $\log(\frac{1}{\epsilon})$.

By Lemma 1, the problem boils down to the following:

Given $\bar{x} \in \mathbb{R}^n$ and $\delta > 0$, find $\bar{u} \in \mathcal{S}^{n-1}$ be such that

$$\mu(H^+(\bar{u}, \bar{x})) \leq \inf_{u \in \mathcal{S}^{n-1}} \mu(H^+(u, \bar{x})) + \delta. \tag{6}$$

Given $\bar{x}$, let $\mathcal{P}_{\bar{x}}$ be the set of all partitions of the vertices of P into two sets that can be achieved by a hyperplane through $\bar{x}$. This induces a covering of the sphere $\mathcal{S}^{n-1}$: For each $X \in \mathcal{P}$ define U_X to be the set of $u \in \mathcal{S}^{n-1}$ such that the hyperplane $u \cdot x = u \cdot \bar{x}$ induces the partition X on the vertices of P. The number of such partitions is closely related to the VC-dimension of hyperplanes, and in particular, is easily seen to be $O(M^n)$ where M is the number of vertices of P. Moreover, one can enumerate these partitions in the same amount of time, by picking $n-1$ vertices $\{v_1, \ldots, v_{n-1}\}$ of P such that $\{\bar{x}, v_1, \ldots, v_{n-1}\}$ are affinely independent.

To solve problem (6), we will proceed along these steps.

1. For each $X \in \mathcal{P}$, find $\bar{u}_X \in \mathcal{S}^{n-1}$ be such that
$$\mu(H^+(\bar{u}, \bar{x})) \leq \inf_{u \in U_X} \mu(H^+(u, \bar{x})) + \delta.$$
2. Pick X^* such that $\mu(H^+(u_{X^*}, \bar{x})) \leq \mu(H^+(u_X, \bar{x}))$ for all $X \in \mathcal{P}$ and report u_{X^*} as the solution to (6).

To complete the proof, we need to implement Step 1. above in polynomial time. This is done in Lemma 3.

Lemma 3. *For a fixed $X \in \mathcal{P}$, one can compute $\bar{u}_X \in \mathcal{S}^{n-1}$ such that*

$$\mu(H^+(\bar{u}, \bar{x})) \leq \inf_{u \in U_X} \mu(H^+(u, \bar{x})) + \delta,$$

using an algorithm whose running time is bounded by a polynomial in $\log(\frac{1}{\delta})$ and the size of an irredundant description of P.

This lemma can be proved using methods from quantifier elimination and the proof appears in the Appendix (Section 6.5).

Counting measure on the integer points in two dimensional polytopes. If we use the counting measure on the integer points in a polytope, the algorithm requires no accuracy parameter ϵ.

Theorem 5. *Let $P = \{x \in \mathbb{R}^2 \mid Ax \leq b\}$ be a rational polytope, where $A \in \mathbb{Z}^{m\times 2}$ and $b \in \mathbb{Z}^m$, such that $P \cap \mathbb{Z}^2 \neq \emptyset$. Let μ denote the uniform measure on $P \cap \mathbb{Z}^2$. Then in polynomial time in the input-size of A and b, one can compute a point*

$$z \in \mathcal{C}(\mathbb{Z}^2, \mu).$$

Proof. As already stressed in the previous section, it suffices to show that for a given $\bar{x} \in \mathbb{Z}$ one can compute in polynomial time

$$\bar{u} := \operatorname*{argmin}_{u \in \mathcal{S}^1} \mu(H^+(u, \bar{x})).$$

Let $g : [0, 2\pi) \mapsto [0, 1]$ be defined as $g(\alpha) := \mu(H^+((\sin(\alpha), \cos(\alpha))^\mathsf{T}, \bar{x}))$. The key observations are that g is piecewise constant and that the domain $[0, 2\pi)$ can be partitioned into a polynomial number of intervals S_i such that g is monotone on each of them. This implies, that in order to compute $\bar{u}$, one only needs to evaluate g at the beginning and the end of each interval S_i.

Let $l^+(\alpha)$ denote the line segment $P \cap \{\bar{x} + \lambda(\sin(\alpha + \pi/2), \cos(\alpha + \pi/2))^\mathsf{T} \mid \lambda \geq 0\}$ and let $l^-(\alpha)$ denote $P \cap \{\bar{x} + \lambda(\sin(\alpha - \pi/2), \cos(\alpha - \pi/2))^\mathsf{T} \mid \lambda \geq 0\}$. Observe that $g(\alpha)$ is monotone increasing if the line segment $l^+(\alpha)$ is longer than the line segment $l^-(\alpha)$ and $g(\alpha)$ is monotone decreasing if the line segment $l^+(\alpha)$ is shorter than the line segment $l^-(\alpha)$. Hence, the monotonicity can only change when the two lengths are equal. All those critical α can be computed by comparing each pair of facets. $\square$

5.2 Approximation algorithms

A Lenstra-type algorithm to compute approximate centerpoints. As we already pointed out in Section 2, centerpoints can be used to design "optimal" oracle-based algorithms for convex mixed-integer optimization problems. In turn, it is possible to employ linear mixed-integer optimization techniques to compute approximate centerpoints. However, this comes with a significant loss in the approximation guarantee.

Theorem 6. *Let $n, d \in \mathbb{N}$ be fixed and let P be a rational polytope. Then in polynomial time in the input-size of P, one can find a point*

$$z \in D_{\mu_{mixed,P}}\left(\frac{1}{2^{n^2}(d+1)^{(n+1)}}\right) \cap (\mathbb{Z}^n \times \mathbb{R}^d).$$

Recall the definition of $\mu_{mixed,P}$ from (3); the proof of Theorem 6 appears in the Appendix (Section 6.6).

Computing approximate centerpoints with a Monte-Carlo algorithm. In this section, we compute ϵ-centerpoints, but for any family of measures from which one can sample uniformly. However, now the algorithm's runtime depends polynomially on $\frac{1}{\epsilon}$, as opposed to $\log(\frac{1}{\epsilon})$ as for the uniform measure on rational polytopes from Section 5.1.

Suppose we have access to two black-box algorithms:

1. OPT is an algorithm which works for some family $\mathcal{S}$ of closed subsets of $\mathbb{R}^n$. OPT takes as input a closed set $S \in \mathcal{S}$ and computes $\mathrm{argmax}_{x \in S}\, g(x)$ for any quasi-concave function g, given an evaluation oracle for g and a separation oracle for the sets $\{x \mid g(x) \geq \alpha\}_{\alpha \in \mathbb{R}}$. Let $T_1(S)$ be the number of calls that OPT makes to the evaluation and separation oracles, and $T_2(S)$ be the number of elementary arithmetic operations OPT makes during its execution.
2. SAMPLE is an algorithm which works for some family of probability measures Γ. SAMPLE takes as input a measure $\mu \in \Gamma$ and produces a sample point $x \in \mathbb{R}^n$ from the measure μ. Let $T(\mu)$ be the running time for SAMPLE.

We now show that with access to the above two algorithms, one can compute an ϵ-centerpoint for $(S, \mu) \in \mathcal{S} \times \Gamma$.

Theorem 7. *Let $\mathcal{S}$ be a family of closed subsets of $\mathbb{R}^n$ equipped with an algorithm OPT as described above, and let Γ be a family of measures on $\mathbb{R}^n$ equipped with an algorithm SAMPLE as described above.*

There exists a Monte Carlo algorithm which takes as input $(S, \mu) \in \mathcal{S} \times \Gamma$, real numbers $\epsilon, \delta > 0$ and computes an ϵ-approximate centerpoint for S, μ with probability at least $1 - \delta$. The running time of this algorithm is $T_1(S) \cdot N^d + T_2(S) + T(\mu) \cdot N$, where $N = O(\frac{1}{\epsilon^2}((n+1) + \log \frac{1}{\delta})$.

To prove this theorem, we will need a a deep result from probability theory that has resulted after a long line of research sparked by the seminal ideas of Vapnik and Chervonenkis [24], and culminated in a result of Talagrand [22]. The following theorem is a rewording of Talagrand's result [22], specialized for function classes with bounded VC-dimension.

Theorem 8. *Let (X, μ) be a probability space. Let $\mathcal{F}$ be a family of functions mapping X to $\{0, 1\}$ and let ν be the VC-dimension of the family $\mathcal{F}$. There* *exists a universal constant C, such that for any $\epsilon, \delta > 0$, if M is a sample of size $C \cdot \frac{1}{\epsilon^2}(\nu + \log \frac{1}{\delta})$ drawn independently from X according to μ, then with probability at least $1 - \delta$, for every function $f \in \mathcal{F}$, $|\frac{|\{x \in M \mid f(x)=1\}|}{|M|} - \mu(\{x \in X \mid f(x) = 1\})| \leq \epsilon$.*

Proof (Theorem 7). We call SAMPLE to create a sample M of size $C \cdot \frac{1}{\epsilon^2}((n+1) + \log \frac{1}{\delta})$ by drawing independently and uniformly at random from S (note that M may contain multiple copies of the same point from S). Since the VC-dimension of the family of half spaces in $\mathbb{R}^n$ is $n+1$, we know from Theorem 8 that with probability at least $1-\delta$, for every half space H^+, $\left|\frac{|H^+ \cap M|}{|M|} - \mu(H^+)\right| \leq \epsilon$. Let μ' be the counting measure on M. Then we obtain that $|f_{\mu'}(x) - f_\mu(x)| \leq \epsilon$ for all $x \in \mathbb{R}^n$. Therefore, any $x^* \in \arg\max_{x \in S} f_{\mu'}(x)$ is an ϵ-centerpoint for S. This can be achieved by calling OPT to compute $x^* \in \arg\max_{x \in S} f_{\mu'}(x)$. For this, we need to exhibit evaluation and separation oracles for $f_{\mu'}$. But notice that, by Lemma 1, this can be accomplished by simply implementing the following procedure: given $x \in R^d$, find the best hyperplane H through x such that $\frac{|H^+ \cap M|}{|M|}$ is minimized. This can be done in time $O(|M|^n)$ by simply enumerating all hyperplanes that contain $n-1$ affinely independent points from M. □

The following result is a consequence.

Theorem 9. *Let $n \geq 1$ and $d \geq 0$ be fixed integers. There exists a Monte Carlo algorithm which takes as input an integer $m \geq 1$, a matrix $A \in \mathbb{R}^{m \times (n+d)}$, a vector $b \in \mathbb{R}^m$, real numbers $\epsilon, \delta > 0$ and returns an ϵ-approximate centerpoint when $S = \mathbb{Z}^n \times \mathbb{R}^d$ and μ is the uniform measure on $\{x \in \mathbb{Z}^n \times \mathbb{R}^d \mid Ax \leq b\}$, with probability $1 - \delta$. The running time of the algorithm is a polynomial in $m, size(A, b), \frac{1}{\epsilon}, \log \frac{1}{\delta}$.*

Proof. By using classical results on maximizing quasi-concave functions over the integer points in a polyhedron [10], OPT can be implemented for the family $\mathcal{S}$ which is the collection of all sets S that can be represented as the set of mixed-integer points in a rational polytope. SAMPLE can be implemented for the family Γ which is the uniform measure on the sets S from $\mathcal{S}$ by adapting a result of Igor Pak [20] on $d = 0$ to $d \geq 1$, using results on computing mixed-integer volumes in polynomial time for fixed dimensions [3]. □

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6 Appendix

6.1 Proof of Lemma 1

Proof. For quasi-concavity, see Proposition 1 in [21], and for upper semicontinuity see Proposition 4 in [21]. For the second part of the lemma, let $x \in \mathbb{R}^n$ be such that $f(x) \geq f(\bar{x}) + \delta$ then

$$\mu(H^+(\bar{u}, x)) \geq f(x) \geq f(\bar{x}) + \delta > f(\bar{x}) + \frac{\delta}{2} \geq \mu(H^+(\bar{u}, \bar{x}))$$

and thus $\mu(H^+(\bar{u}, x)) > \mu(H^+(\bar{u}, \bar{x}))$. This implies $\bar{u} \cdot x < \bar{u} \cdot \bar{x}$. □

6.2 Proof of Theorem 2

Proof. The proof follows along similar lines as [21, Proposition 9]. It suffices to show that for every $\epsilon > 0$, the set $D_\mu(h(S)^{-1} - \epsilon) \cap S$ is nonempty. By standard measure-theoretic arguments, there exists a ball B centered at the origin such that $\mu(B) \geq 1 - \frac{\epsilon}{2}$ and $D_\mu(h(S)^{-1} - \epsilon) \subseteq B$ (by Remark 1, $D_\mu(h(S)^{-1} - \epsilon)$ is compact). By Proposition 6 in [21],

$$D_\mu(h(S)^{-1} - \epsilon) \cap S = \bigcap \{H \cap S \mid H \text{ is a closed half space with } \mu(H) \geq 1 - (h(S)^{-1} - \epsilon)\}.$$

Define $\mathcal{C} = \{B \cap H \cap S \mid H \text{ is a closed half space with } \mu(H) \geq 1 - (h(S)^{-1} - \epsilon)\}$. Thus, $\mathcal{C}$ is a family of compact sets such that $D_\mu(h(S)^{-1} - \epsilon) \cap S = \bigcap \{C \mid C \in \mathcal{C}\}$. For any subset $\{C_1, \ldots, C_{h(S)}\} \subseteq \mathcal{C}$ of size $h(S)$, we claim

$$\mu(C_1^c \cup \ldots C_{h(S)}^c) \leq 1 - h(S)\frac{\epsilon}{2}.$$

This is because each $C_i^c = B^c \cup H_i^c \cup S^c$ for some half space H_i satisfying $\mu(H_i^c) \leq h(S)^{-1} - \epsilon$. Since $\mu(B^c) \leq \frac{\epsilon}{2}$ and $\mu(S^c) = 0$, we obtain that $\mu(C_i^c) \leq h(S)^{-1} - \frac{\epsilon}{2}$. Therefore,

$$\mu(C_1 \cap \ldots \cap C_{h(S)}) = 1 - (\mu(C_1^c \cup \ldots C_{h(S)}^c)) \geq 1 - (1 - h(S)\frac{\epsilon}{2}) = h(S)\frac{\epsilon}{2} > 0.$$

This implies that $C_1 \cap \ldots \cap C_{h(S)} \neq \emptyset$. Therefore, by definition of $h(S)$, for every finite subset $\{C_1, \ldots, C_m\} \subseteq \mathcal{C}$, $C_1 \cap \ldots \cap C_m \neq \emptyset$. By the finite intersection property of compact sets, we obtain that $D_\mu(h(S)^{-1} - \epsilon) \cap S = \bigcap \{C \mid C \in \mathcal{C}\}$ is nonempty. □

6.3 Proof of Proposition 1

Proof. We want to derive a contradiction. Therefore, we assume that there exist at least two distinct centerpoints $x^1, x^2 \in \mathcal{C}(\mathbb{R}^n, \mu)$ such that $x^1 \neq x^2$. Let $x^0 = \frac{1}{2}(x^1 + x^2)$. By Lemma 1 there exists a $u \in \mathcal{S}^{n-1}$ such that $\tau(u, x^i) = \mathcal{F}(\mathbb{R}^n, \mu)$ for $i = 0, 1, 2$. W.l.o.g. we may assume that $x^1 = e_1$, $x^2 = -e_1$ and $u = e_2$, where e_i, with $i \in \{1, \ldots, n\}$, denotes the i-th unit-vector in $\mathbb{R}^n$.

The idea will be to consider a perturbation $\tilde{u}$ of u, such that $x^1 \in \text{int}(H^+(\tilde{u}, 0))$ and $x^2 \notin H^+(\tilde{u}, 0)$. Then, we evaluate the measure of $H^+(\tilde{u}, x^1)$, i.e. the volume of

$P \cap H^+(\tilde{u}, x^1)$, with respect to the point 0 and show that the measure decreases. Let $H^-(u,x) := \{y \in \mathbb{R}^n \mid u^\mathsf{T}(y-x) \le 0\}$. Then, more precisely, we show that

$$\begin{aligned} \mu(H^+(\tilde{u}, x^1)) - \mu(H^+(u,0)) = \; & \mu(H^+(\tilde{u},0) \cap H^-(u,0)) \\ & - \mu(H^-(\tilde{u},0) \cap H^+(u,0)) \\ & - \mu(H^+(\tilde{u},0) \cap H^-(\tilde{u},x^1)) < 0, \end{aligned} \tag{7}$$

contradicting that x^1 is a centerpoint.

We change from the Cartesian coordinate system to a cylindrical coordinate system. We define the variables $(\rho, \phi, y_1, \ldots, y_{n-2})$ as follows

$$x_1 = \rho\cos(\phi), \;\; x_2 = \rho\sin(\phi) \;\text{ and }\; x_i = y_{i-2} \text{ for } i = 3, \ldots, .n.$$

It is well known that the determinant of the corresponding Jacobian matrix is ρ.

Let $D := \{(\phi, y) \in [0, 2\pi) \times \mathbb{R}^{n-2} \mid \exists \rho \in \mathbb{R}_+ \text{ such that } (\rho, \phi, y) \in S\}$. Note that the set D is a closed set in a normal topological space. We define the functions $R, r : [0, 2\pi) \times \mathbb{R}^{n-2} \to \mathbb{R}_+$ as follows. For $(\phi, y) \in D$ we define

$$R(\phi, y) := \max\{\rho \in \mathbb{R}_+ \mid (\rho, \phi, y) \in P\}$$

and

$$r(\phi, y) := \min\{\rho \in \mathbb{R}_+ \mid (\rho, \phi, y) \in P\}.$$

Note that R is continuous on D. Hence, by Tietze's Extension Theorem [8, Theorem 5.1], we extend R continuously for $(\phi, y) \notin D$. Further, for $(\phi, y) \notin D$, we define $r(\phi, y) := R(\phi, y)$. We can now, for example, express $\mu(H^+(u,0))$ as follows

$$\int_{\mathbb{R}^{n-2}} \int_0^\pi \int_{r(\phi,y)}^{R(\phi,y)} \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}y.$$

A crucial observation is that since 0 is a centerpoint we have that

$$\int_{\mathbb{R}^{n-2}} R(0,y)^2 - r(0,y)^2 \, \mathrm{d}y = \int_{\mathbb{R}^{n-2}} R(\pi,y)^2 - r(\pi,y)^2 \, \mathrm{d}y. \tag{8}$$

While R is continuous, the function r is only almost everywhere continuous. Since P is a polytope, r has only finitely many discontinuities in ϕ for any fixed y. Further, note that if r is not continuous almost everywhere for variable ϕ and y, this would imply that the a set of discontinuity points with non zero measure correspond to one facet of P. This in turn would imply that the centerpoint would also lie on the boundary, a contradiction.

Next we want to approximate the terms in (7). For that, let $\epsilon > 0$. Since R and r are continuous for points corresponding to the interior of P, there exists a pertubation $\sigma > 0$ such that for almost all $(\phi, y) \in [0, \sigma] \times \mathbb{R}^{n-2}$ we have $|R(\phi,y) - R(0,y)|, \; |r(\phi,y) - r(0,y)|, \; |R(\pi+\phi,y) - R(\pi,y)|, \; |r(\pi+\phi,y) - r(\pi,y)| \le \epsilon$.

We bound $\mu(H^+(\tilde{u},0) \cap H^-(u,0))$ from above by

$$\int_{\mathbb{R}^{n-2}} \int_0^\sigma \int_{r(\phi,y)}^{R(\phi,y)} \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}y \le \int_{\mathbb{R}^{n-2}} \sigma[(R(0,y)+\epsilon)^2 - (r(0,y)-\epsilon)^2] \, \mathrm{d}y,$$

$\mu(H^-(\tilde{u},0) \cap H^+(u,0))$ from below by

$$\int_{\mathbb{R}^{n-2}} \sigma[(R(\pi,y)-\epsilon)^2 - (r(\pi,y)+\epsilon)^2] \, \mathrm{d}y,$$

and finally $\mu(H^+(\tilde{u},0)\cap H^-(\tilde{u},x^1)) \leq \mu(\text{conv}(\{x^1\},\ H(\tilde{u},0)))$ which we can approximate again from below by

$$\int_{\mathbb{R}^{n-2}} \sin(\sigma)[R(0,y)-r(0,y)-2\epsilon]\,\mathrm{d}y.$$

In order to prove (7), it remains to show that

$$\begin{aligned} &\sin(\sigma)\int_{\mathbb{R}^{n-2}}[R(0,y)-r(0,y)-2\epsilon]\,\mathrm{d}y \\ >&\sigma\int_{\mathbb{R}^{n-2}}[(R(0,y)+\epsilon)^2-(r(0,y)-\epsilon)^2]-[(R(\pi,y)-\epsilon)^2-(r(\pi,y)+\epsilon)^2]\,\mathrm{d}y. \end{aligned} \tag{9}$$

For small σ we may assume that $\sin(\sigma)=\sigma$. Together with (8) we can simplify the above inequation to

$$\int_{\mathbb{R}^{n-2}}[R(0,y)-r(0,y)-2\epsilon]\,\mathrm{d}y > \int_{\mathbb{R}^{n-2}} 2\epsilon[R(0,y)+r(0,y)+R(\pi,y)+r(\pi,y)]\,\mathrm{d}y.$$

□

6.4 Proof of Lemma 2

For the proof of Lemma 2 we need two technical auxiliary lemmata. The first lemma gives an ellipsoidal approximation. For the proof of Lemma 2 the classical John Ellipsoid would suffice and this would improve the bound on the lattice-width, however in view Theorem 3 we develop an ellipsoidal approximation with respect to centerpoints. In the second lemma we extend the ellipsoidal approximation and, in a certain sense, introduce a reduced basis for mixed integer lattices.

Lemma 4. *Let $K\subset\mathbb{R}^n$ be a compact convex set with nonempty interior and let μ be the uniform measure with respect to K. Further let $x^\star\in\mathcal{C}(\mathbb{R}^n,\mu)$. Then, there exists an ellipsoid E such that*

$$x^\star+E\subset K\subset x^\star+n^{7/2}E.$$

Proof. Without loss of generality we assume that $x^\star=0$. Let E' be the maximum volume ellipsoid contained in $K-K$. Then, by John's characterization of inscribed ellipsoids of maximum volume for centrally symmetric convex bodies it holds that $E'\subset K-K\subset\sqrt{n}E'$ (see [11, Section 11.1]).

We prove that for any $u\in S^{n-1}$

$$\frac{1}{n^3}\leq\left|\frac{\max_{x\in K}u^\mathsf{T}x}{\min_{x\in K}u^\mathsf{T}x}\right|\leq n^3.$$

Without loss of generality we also assume that $\min_{x\in K}u^\mathsf{T}x=-1$ and $\max_{x\in K}u^\mathsf{T}x=:\alpha$. where $u=e_1$, i.e. u is equal to the first unit vector. To derive a contradiction, assume that $\alpha<\frac{1}{n^3}$. Let $z:=\text{argmax}_{x\in K}u^\mathsf{T}x$. We define for every $t\in\mathbb{R}$ the set $K_t:=K\cap\{x\in\mathbb{R}^n \mid u^\mathsf{T}x=t\}$. Further, we define $C:=z+\text{cone}(K_0-z)$, $X_1:=\{x\in\mathbb{R}^n \mid -1\leq u^\mathsf{T}x\leq 0\}$ and $X_2=\{x\in\mathbb{R}^n \mid 0\leq u^\mathsf{T}x\leq a\}$. Then $K\cap X_1\subset C\cap X_1$ and $K\cap X_2\supset C\cap X_2$. By Grünbaum's theorem [12, Theorem 2] we have that $1-\left(\frac{n}{n+1}\right)^n\geq\mu(K\cap X_1)\geq\mu(C\cap X_1)=\frac{1}{n}V$, where V represents the low dimensional measure of K_0. On the other hand we have $\left(\frac{n}{n+1}\right)^n\leq\mu(K\cap X_2)\leq\mu(C\cap X_2)=(\frac{(1+\alpha)^n}{n}-\frac{1}{n})V$. Combining these two inequalities results in a contradiction for all $n\geq 2$. Now, it follows that $K\subset\frac{n^3}{n^3+1}\sqrt{n}E'$ and $\frac{1}{n^3+1}E'\subset K$. The lemma follows by choosing $E=\frac{1}{n^3+1}E'$. □

Lemma 5. *Let $K \subset \mathbb{R}^{n+d}$ be a compact convex set with nonempty interior and let μ be the uniform measure with respect to K. Further, let $x^\star \in \mathcal{C}(\mathbb{R}^n, \mu)$. If $\omega(K|_{\mathbb{R}^n}) \geq cn(n+d)^{7/2}\alpha n^n\sqrt{n}$ for some $c \in \mathbb{Z}_+$, then there exists a matrix $B = [b_1, \ldots, b_n] \in \mathbb{R}^{(n+d)\times n}$ such that*

$$x^\star + cB[-1/2, 1/2]^n \subset K$$

and $b_1|_{\mathbb{R}^n}, \ldots, b_n|_{\mathbb{R}^n}$ is a Korkine-Zolotarev basis [15] of $\mathbb{Z}^n$ with respect to the maximum inscribed ellipsoid of $K|_{\mathbb{R}^n}$.

Proof. Without loss of generality we assume that $x^\star = 0$. Then, by Lemma 4, there exists an ellipsoid E such that $E \subset K \subset (n+d)^{7/2}E$. We define $\phi : \mathbb{R}^n \to \mathbb{R}^n$ as the affine map such that $\phi(E|_{\mathbb{R}^n}) = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$. Then, let $K' := \phi(K|_{\mathbb{R}^n})$ and $\Lambda := \phi(\mathbb{Z}^n)$.

Let B denote a Korkine-Zolotarev basis of Λ [15]. Then, a well known property is that

$$\|B_{\star,1}\|_2 \cdots \|B_{\star,n}\|_2 \leq \alpha n^n \det(\Lambda) \tag{10}$$

(see [16, Theorem 2.3]), where α is a universal constant and $B_{\star,i}$ denotes the i-th column of B for $i = 1, \ldots, n$. Further note that for the Gram-Schmidt orthogonalization $\tilde{B}_{\star,1}, \ldots, \tilde{B}_{\star,n}$ of B it holds that $\|\tilde{B}_{\star,i}\|_2 \leq \|B_{\star,i}\|_2$ for all $i = 1, \ldots, n$ and it holds that $\det(\Lambda) = \det(B) = \det(\tilde{B}) = \prod_{i=1}^n \|\tilde{B}_{\star,i}\|_2$ (see for example [11, Chapter 28]). Since $K \subset (n+d)^{7/2}E$, the definition of the lattice-width implies that $\|\tilde{B}_{\star,n}\|_2 \leq \frac{2(n+d)^{7/2}}{\omega}$. Together with (10) this gives

$$\|B_{\star,n}\|_2 \leq \alpha n^n \frac{2(n+d)^{7/2}}{\omega}.$$

By changing the indices in the Gram-Schmidt orthogonalization we obtain a bound on the Euclidean length of all Korkine-Zolotarev vectors, i.e., for all $i = 1, \ldots, n$

$$\|B_{\star,i}\|_2 \leq \alpha n^n \frac{2(n+d)^{7/2}}{\omega}.$$

In turn this implies that

$$C := \frac{1}{\sqrt{n}} \frac{1}{\alpha n^n} \frac{\omega}{(n+d)^{7/2}} B[-1/2, 1/2]^n \subset \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\} \subset K'.$$

Since E is centrally symmetric there exists a full-dimensional parallelotope $P = c \cdot \sum_{i=1}^n \mathrm{conv}(-b_i, b_i)$ contained in K such that $P|_{\mathbb{R}^n} = \phi^{-1}(C)$ and, in particular, $b_i|_{\mathbb{R}^n} = B_{\star,i}$. □

We are now ready to prove Lemma 2.

Proof (Lemma 2). By Lemma 5 there exists a matrix $B \in \mathbb{R}^{(n+d)\times n}$ such that $x^\star + cn\, B[-1/2, 1/2]^n \subset K$. Since $B|_{\mathbb{R}}$ is unimodular, we may assume after an unimodular transformation that $B|_{\mathbb{R}}$ equals the identity matrix. After a further volume preserving linear transformation we may even assume that B equals the first n unite vectors. Since K has full volume, there exists an $\epsilon > 0$ such that $c\,n[-1/2, 1/2]^n \times \epsilon[-1/2, 1/2]^d \subset K$.

In the next step, we want to exploit that $\lim_{k\to\infty} \frac{1}{k^d}|K \cap (\mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d)| = \bar{\nu}(K \cap (\mathbb{Z}^n \times \mathbb{R}^d))$, where $|\cdot|$ denotes the cardinality. For that we prove the following claim: Let $P = c_n[-1/2, 1/2]^n \times \epsilon k[-1/2k, 1/2k]^d \subset K$. Then

$$\left(1 - \frac{1}{c_n}\right)^n \left(1 - \frac{1}{\epsilon k}\right)^d \leq \frac{\frac{1}{k^d}|K \cap (\mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d)|}{\bar{\mu}(K)} \leq \left(1 + \frac{1}{c_n}\right)^n \left(1 + \frac{1}{\epsilon k}\right)^d.$$

Let $Q := [-1/2, 1/2]^n \times [-1/2k, 1/2k]^d$. Further we define $\bar{K} := (K \cap (\mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d)) + Q$ and $K_{\lambda,\gamma} := \begin{pmatrix} (1+\lambda)I_n & 0 \\ 0 & (1+\gamma)I_d \end{pmatrix} K$ with $\lambda, \gamma \in \mathbb{R}$ and $\lambda, \gamma > -1$. Then $\frac{1}{k^d}|K \cap (\mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d)| = \bar{\mu}(\bar{K}) \leq \bar{\mu}(K + Q)$. It remains to observe that $K + Q \subset K_{1/c_n, 1/\epsilon k}$. Hence $\frac{1}{k^d}|K \cap (\mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d)| = \bar{\mu}(\bar{K}) \leq \bar{\mu}(K_{1/c_n,1/\epsilon k}) = (1 + 1/c_n)^n (1 + 1/\epsilon k)^d \bar{\mu}(K)$. In order to prove the lower bound assume that there exists an $x \in K_{-1/c_n, -1/\epsilon k} \setminus \bar{K}$. We define $z \in \mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d$, such that $z_i = \lfloor x_i \rceil$ if $i \leq n$ and $z_i = \frac{1}{k}\lfloor k\, x_i \rceil$ else.[3] Then, since $x - z \in K_{1/c_n - 1, 1/\epsilon k - 1}$ the point z must be in $K \cap (\mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d)$. This contradicts $x \notin z + B \subset \bar{K}$. Hence $\frac{1}{k^d}|K \cap (\mathbb{Z}^n \times \frac{1}{k}\mathbb{Z}^d)| = \bar{\mu}(\bar{K}) \geq \bar{\mu}(K_{-1/c_n, -1/\epsilon k}) = (1 - 1/c_n)^n (1 - 1/\epsilon k)^d \bar{\mu}(K)$.

By the choice of the bound on the lattice-width we can set $c_n = cn$ and then by taking the limit, with k going to infinity, we derive the lemma. $\square$

6.5 Proof of Lemma 3

Proof. For a fixed partition $X \in \mathcal{P}$ the feasible region U_X is described by linear inequalities and a single quadratic inequality $u_1^2 + u_2^2 + \ldots + u_n^2 = 1$. We claim the objective function can be written as the ratio of two polynomials in $u_1, \ldots, u_n$. Subject to these constraints, we need to minimize $\mu(H^+(u, \bar{x}))$. Since X is the partition of the vertices of P induced by the hyperplane $u \cdot x = u \cdot \bar{x}$ (since $u \in U_X$), the set of edges intersected by this hyperplane is fixed. Moreover, the coordinates of the point of intersection of any such edge and this hyperplane can be expressed by a ratio of linear functions of $u_1, \ldots, u_n$. Indeed, suppose the edge intersected is the convex hull of the vertices $v_1, v_2 \in \mathbb{R}^n$. Then there exists $\lambda \in [0, 1]$ such that $u \cdot (\lambda v_1 + (1 - \lambda)v_2) = u \cdot \bar{x}$. Thus, $\lambda = \frac{u \cdot (\bar{x} - v_2)}{u \cdot (v_2 - v_1)}$, and the point of intersection is $\lambda v_1 + (1 - \lambda)v_2$ which is a ratio of linear functions of u. Also, $P \cap H^+(u, \bar{x})$ can be decomposed into a simplicial complex whose combinatorial structure only depends on X and not on the actual values of $u \in U_X$. The volume of $P \cap H^+(u, \bar{x}))$ can be written as the sum of the volumes of these simplices. Since the volume of a simplex can be written as a polynomial in the coordinates of its vertices, we obtain that $\mu(H^+(u, \bar{x}))$ is sum of ratios of polynomials in $u_1, \ldots, u_n$ with degree bounded by a function of n only, which can be written as a single ratio of polynomials in $u_1, \ldots, u_n$ where the degrees of the polynomials are bounded by a function of n only. Thus, finding $u_X \in \operatorname{argmin}_{u \in U_X} \mu(H^+(u, \bar{x}))$ is equivalent to solving a mathematical optimization problem of the following type:

$$\min_{u_1,\ldots,u_n} \frac{p(u)}{q(u)} \qquad \text{s.t. } A \cdot u \leq b,\ \ u_1^2 + u_2^2 + \ldots + u_n^2 = 1.$$

The above is equivalent to the following polynomial optimization problem:

$$\min_{z,u_1,\ldots u_n} z \qquad \text{s.t. } p(u) = z \cdot q(u),\ \ A \cdot u \leq b,\ \ u_1^2 + u_2^2 + \ldots + u_n^2 = 1.$$

This optimization problem can be solved to within δ accuracy by performing a binary search on the objective value and using quantifier elimination methods for testing feasibility of polynomial systems of inequalities and equalities. For polynomial systems with a fixed number of variables this can be done in polynomial time in the size of the coefficients [4]. $\square$

[3] For $x \in \mathbb{R}^n$ we denote With $\lfloor x \rceil$ the point $z \in \mathbb{Z}^n$ such that for each component $-\frac{1}{2} < z_i - x_i \leq \frac{1}{2}$.

6.6 Proof of Theorem 6

Proof. By Theorem 4, the statement holds for $n = 0$. Also, since Theorem 3 is constructive, there exists a $\bar{\omega}$ that only depends on n and d, such that the theorem holds true provided that the lattice-width of P is larger than $\bar{\omega}$.

By induction we assume that the result is true for $n-1$. Further, we may assume that the lattice width is smaller than $\bar{\omega}$. Without loss of generality, we assume that the flatness direction of P is equal to n-th unit vector, i.e., $\min_{x\in P} x_n \geq 0$ and $\max_{x\in P} x_n \leq \bar{\omega}$. We define $P_i := P \cap \{x \in \mathbb{R}^{n+d} \mid x_n = i\}$ and the corresponding uniform measure μ_i. By the induction hypothesis, we can compute $z_i \in D_{\mu_i}(2^{-(n-1)^2}(d+1)^{-n}) \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ for $i = 0, \dots, \bar{\omega}$.

We define the finite auxiliary measure:

$$\bar{\mu}(x) := \begin{cases} \mu(P_i) & \text{if } x = z_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, with a brute force approach, we compute the centerpoint z in $\mathcal{C}(\mathbb{Z}^n \times \mathbb{R}^d, \bar{\mu})$.

It remains to show that $z \in D_\mu(\frac{1}{2^{n^2}(d+1)^{n+1}})$. Let H^+ be any half-space containing z. Note that, for all i we have $\mu(P_i \cap H^+) \geq \frac{1}{2^{(n-1)^2}(d+1)^n}\bar{\mu}(\{z_i\} \cap H^+)$. Hence,

$$\mu(P \cap H^+) = \sum_{i=0}^{\bar{\omega}} \mu(P_i \cap H^+) \geq \frac{1}{2^{(n-1)^2}(d+1)^n} \sum_{i=0}^{\bar{\omega}} \bar{\mu}(\{z_i\} \cap H^+) \geq \frac{1}{2^{n^2}(d+1)^{n+1}},$$

where the last inequality comes from Theorem 2. $\square$