

1. Compute the following integrals (note that some of them are indefinite integrals and some are definite integrals).

(a) (6 pts) $\int \frac{e^{7x}}{50+e^{7x}} dx$. Substitute $u = 50 + e^{7x}$
 $du = e^{7x}(7) dx$
 $\Rightarrow \frac{1}{7} du = e^{7x} dx$

$$\begin{aligned} \int \frac{e^{7x}}{50+e^{7x}} dx &= \int \frac{\left(\frac{1}{7} du\right)}{u} \\ &= \frac{1}{7} \int \frac{1}{u} du = \frac{1}{7} \ln|u| = \boxed{\frac{1}{7} \ln|50+e^{7x}| + C} \end{aligned}$$

(b) (7 pts) $\int x\sqrt{5-x} dx$. Substitute $u = 5-x$
 $du = -dx, x = 5-u$

$$\begin{aligned} \Rightarrow \int x\sqrt{5-x} dx &= \int (5-u)\sqrt{u} (-du) \\ &= - \int (5\sqrt{u} - u^{3/2}) du = - \frac{5u^{3/2}}{\frac{3}{2}} + \frac{u^{5/2}}{\frac{5}{2}} \\ &= \boxed{-\frac{5}{3}(5-x)^{3/2} + \frac{2}{5}(5-x)^{5/2}} \end{aligned}$$

(c) (7 pts) $\int x \ln(x+1) dx$.

use integration by parts $g(x) = x, f(x) = \ln(x+1)$

$$\int x \ln(x+1) dx = \bullet \ln(x+1) \frac{x^2}{2} \bullet - \int \frac{1}{x+1} \frac{x^2}{2} dx$$

To evaluate,

$$\int \frac{x^2}{2(x+1)} dx \quad \text{use } \cancel{\text{partial }} \text{ long division :}$$

$$\frac{x^2}{x+1} = x-1 + \frac{1}{x+1}$$

$$\Rightarrow \int \frac{x^2}{2(x+1)} dx = \frac{1}{2} \left[\int \left((x-1) + \frac{1}{x+1} \right) dx \right] = \frac{1}{2} \left(\frac{x^2}{2} - x + \ln|x+1| \right)$$

Final answer: $\boxed{\int \ln(x+1) \frac{x^2}{2} dx - \frac{1}{2} \left(\frac{x^2}{2} - x + \ln|x+1| \right) + C}$

$$(d) (7 \text{ pts}) \int (\cos(x) + \sec(x))^2 dx.$$

$$\begin{aligned} \int (\cos^2 x + \sec^2 x)^2 dx &= \int \cos^2 x + 2\cos(x)\sec(x) + \sec^2(x) dx \\ &= \int \cos^2 x dx + \int 2\cos(x)\sec(x) dx + \int \sec^2(x) dx \\ &= \int \frac{1 + \cos 2x}{2} dx + \int 2 dx + \int \sec^2(x) dx = \boxed{\frac{x}{2} + \frac{\sin 2x}{4} + 2x + \tan x + C} \end{aligned}$$

$$(e) (7 \text{ pts}) \int \tan^3(x) dx.$$

$$\begin{aligned} \tan^2 x &= \sec^2 x - 1 \\ \int \tan^3 x dx &= \int (\tan(x)(\sec^2 x)) dx = \int \tan(x)(\sec^2(x) - 1) dx \\ &= \int \tan(x) \sec^2(x) dx - \int \tan(x) dx = \frac{\tan^2 x}{2} + \ln |\cos(x)| + C \end{aligned}$$

$$(f) (7 \text{ pts}) \int (\ln(x))^2 dx.$$

use integration by parts, $f(x) = (\ln x)^2$ $g(x) = 1$

$$\begin{aligned} \int (\ln x)^2 dx &= (\ln x)^2 x - \int 2(\ln x) \frac{1}{x} \cdot x dx \\ &= (\ln x)^2 x - 2 \int \ln x dx \end{aligned}$$

use integration by parts again $f(x) = (\ln x)$ $g(x) = 1$

$$\int \ln x dx = (\ln x)x - \int \frac{1}{x} \cdot x dx = (\ln x)x - x$$

$$\therefore \int (\ln x)^2 dx = \boxed{(\ln x)^2 x - 2(\ln x)x + x + C}$$

$$(g) (7 \text{ pts}) \int \frac{1+x}{x+3e^{-x}} dx.$$

Multiply the numerator & denominator by e^x .

$$\int \frac{1+x}{x+3e^{-x}} dx = \int \frac{e^x(1+x)}{e^x(x+3e^{-x})} dx = \int \frac{e^x + xe^x}{xe^x + 3} dx$$

$$\text{Substitute } u = xe^x + 3, \quad du = (e^x + xe^x) dx$$

$$\int \frac{e^x + xe^x}{xe^x + 3} dx = \int \frac{du}{u} = \ln|u| = \boxed{\ln|e^x + 3| + C}$$

$$(h) (7 \text{ pts}) \int_0^\infty \frac{x^2}{e^{x^3}} dx$$

$$\text{Substitute } u = \cancel{x^3} \quad du = 3x^2 dx$$

$$\int_0^\infty \frac{x^2}{e^{x^3}} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{x^2}{e^{x^3}} dx$$

$$\int \frac{x^2}{e^{x^3}} dx = \int \frac{1}{3} \frac{dx}{e^u} \quad \cancel{du} = \int \frac{1}{3} e^{-u} du = \frac{1}{3} (-e^{-u})$$

$$= -\frac{1}{3} e^{-x^3}$$

$$\therefore \int_0^b \frac{x^2}{e^{x^3}} dx = -\frac{1}{3} e^{-x^3} \Big|_0^b = -\frac{1}{3} e^{-b^3} - \left(-\frac{1}{3} e^0\right) = \frac{1}{3} - \frac{1}{3} e^{-b^3}$$

$$\lim_{b \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{3} e^{-b^3} \right) = \boxed{\frac{1}{3}}$$

2. (10 pts) Consider the region bounded by the curves $y = 3x^3 - x^2 - 10x$ and $y = -x^2 + 2x$. Set up (but DO NOT EVALUATE) the integral(s) to compute the area of this region.

Find pts. of intersection:

$$3x^3 - x^2 - 10x = -x^2 + 2x$$

$$\Rightarrow 3x^3 - 12x = 0 \Rightarrow 3x(x+2)(x-2) = 0$$

$$x = -2, 0, 2.$$

In $[-2, 0]$ $3x^3 - x^2 - 10x$ is above $-x^2 + 2x$
 and in $[0, 2]$ $-x^2 + 2x$ is above.

so area =

$$\int_{-2}^0 (3x^3 - x^2 - 10x) - (-x^2 + 2x) dx + \int_0^2 (-x^2 + 2x) - (3x^3 - x^2 - 10x) dx$$

3. (7 pts) Find $y' = \frac{dy}{dx}$ for $e^{xy} = y^3 + x^2 + \ln(y)$.

use implicit differentiation :

$$\frac{d}{dx} (e^{xy} = y^3 + x^2 + \ln(y))$$

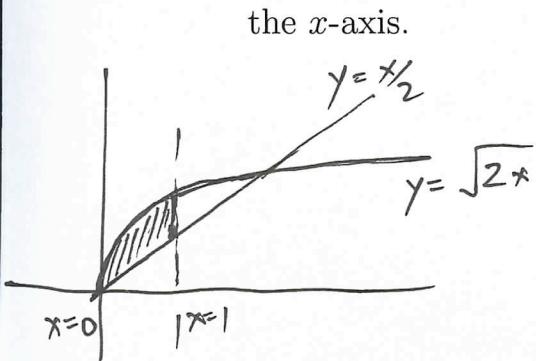
$$\Rightarrow e^{xy} \cdot \frac{d}{dx}(xy) = 3y^2 \frac{dy}{dx} + 2x + \frac{1}{y} \frac{dy}{dx}$$

$$\Rightarrow e^{xy} \left(y + x \frac{dy}{dx} \right) = 3y^2 \frac{dy}{dx} + 2x + \frac{1}{y} \frac{dy}{dx}$$

$$\Rightarrow e^{xy} y - 2x = \left(3y^2 + \frac{1}{y} - xe^{xy} \right) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{e^{xy} y - 2x}{3y^2 + \frac{1}{y} - xe^{xy}}}$$

4. (8 pts) Consider the region bounded by the graphs $y = \sqrt{2x}$, $y = \frac{x}{2}$, $x = 0$ and $x = 1$. Set up (but DO NOT EVALUATE) the integral to compute the volume of the solid obtained by revolving this region around the x -axis.



~~sketch~~

$$\begin{aligned} & \pi \int_0^1 \left[(\sqrt{2x})^2 - \left(\frac{x}{2}\right)^2 \right] dx \\ &= \boxed{\pi \int_0^1 \left(2x - \frac{x^2}{4} \right) dx} \end{aligned}$$

5. (10 pts) Use the Trapezoidal rule with $n = 4$ to estimate the value of the following integral.

$$\int_{-1}^1 \log_{10}(3 + 2x) dx$$

$$n=4 \quad \text{Endpts. : } -1, -0.5, 0, 0.5, 1.$$

$$\text{Trapezoid Rule : } \frac{b-a}{2n} \left[f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5) \right]$$

$$= \frac{1-(-1)}{2 \cdot 4} \left[\log_{10}(3-2) + 2\log_{10}(3-1) + 2\log_{10}3 + 2\log_{10}4 + \log_{10}5 \right]$$

$$= \frac{2}{8} \left(0 + 2\log_{10}2 + 2\log_{10}3 + 2\log_{10}4 + \log_{10}5 \right)$$

$$= 0.25(0.602 + 0.954 + 1.204 + 0.699)$$

$$= \boxed{0.86475}$$

6. (10 pts) The number of wild hogs in a game preserve after t years is given by

$$N(t) = 500 - \frac{400}{1+2t}$$

for $t \geq 0$. What is the average number of wild hogs from $t = 0$ years to $t = 2$ years?

$$\begin{aligned} \text{Avg} &= \frac{1}{2-0} \int_0^2 \left(500 - \frac{400}{1+2t} \right) dt \\ &= \frac{1}{2} \left[500t - \frac{400}{2} \ln|1+2t| \right]_0^2 = \frac{1}{2} (500 \cdot 2 - 200 \ln|5|) \\ &= \boxed{339.056} \end{aligned}$$

7. Let $f(x) = \frac{1}{4\sqrt{x}}$, $1 \leq x \leq 9$ be the probability density function for a continuous random variable measuring the number of hours in a week that Davis commutes spend driving their automobiles.

- (a) (5 pts) Verify that $f(x)$ is a probability density function.

$$\int_1^9 \frac{1}{4\sqrt{x}} dx = \frac{1}{4} \left[\frac{\sqrt{x}}{\frac{1}{2}} \right]_1^9 = \frac{1}{2} (3-1) = \underline{\underline{1}}$$

- (b) (3 pts) Compute the probability that x takes values greater than or equal to 6.

$$\begin{aligned} P(x \geq 6) &= \int_6^9 \frac{1}{4\sqrt{x}} dx = \frac{1}{2} \left[\frac{\sqrt{x}}{\frac{1}{2}} \right]_6^9 = \frac{1}{2} (3 - \cancel{\sqrt{6}}) \\ &= \boxed{0.2752} \end{aligned}$$

(c) (6 pts) Compute the expected value $E[x]$.

$$\int_1^9 x \frac{1}{4\sqrt{x}} dx = \frac{1}{4} \left(\frac{x^{3/2}}{\frac{3}{2}} \right) \Big|_1^9 = \frac{1}{6} (9^{3/2} - 1^{3/2}) \\ = \frac{1}{6} (26) = \underline{\underline{4.3333}}$$

(d) (6 pts) Compute the median for this random variable.

$$\int_1^c \frac{1}{4\sqrt{x}} dx = 0.5 \Rightarrow \frac{1}{2} (\sqrt{c} - 1) = 0.5 \\ \Rightarrow \sqrt{c} = 1 + 2 * 0.5 \\ \Rightarrow c = \boxed{4}$$

8. (10 pts) Assume that while chewing Bubble gum, the amount of sugar in the gum decreases following an exponential decay model. If after 1 minute of chewing, 75% of the original mount of sugar is left, what percent of the original amount of sugar will remain after 10 minutes?

Let y be the amount of sugar, so $y = Ce^{kt}$

at $t=0$, $y=C$.

at $t=1$, $y=0.75C$

$$\Rightarrow 0.75C = Ce^{k(1)} \Rightarrow 0.75 = e^k \Rightarrow k = \ln(0.75)$$

$$\text{at } t=10, y = Ce^{10\ln(0.75)} = (0.0563)C$$

$\therefore \boxed{5.63\% \text{ of the original amount is left}}$

9. (10 pts) Suppose that the area bounded under the curve $y = 10e^x$ and $y = 0$ and between $x = 0$ and $x = b$ is equal to 10. What is the value of b ?

$$\text{Area} = \int_0^b 10e^x dx = 10e^x \Big|_0^b = 10(e^b - e^0) \\ = 10(e^b - 1)$$

$$\therefore 10 = 10(e^b - 1)$$

$$\Rightarrow e^b - 1 = 1 \Rightarrow e^b = 2$$

$$\Rightarrow \boxed{b = \ln 2}$$

10. (10pts) We are given a probability density function $f(x) = kx^8$ for a continuous random variable with the range of values $0 \leq x \leq b$. It is known that the median for this random variable is 1. What is the value of k and b ?

$$\text{Median} = 1$$

$$\therefore \int_0^1 kx^8 dx = 0.5$$

$$\Rightarrow k \frac{x^9}{9} \Big|_0^1 = 0.5 \Rightarrow k \left(\frac{1}{9}\right) = 0.5$$

$$\Rightarrow k = \boxed{4.5}$$

Moreover, $\int_0^b f(x) dx = 1 \Rightarrow \int_0^b kx^8 dx = 1$

$$= \int_0^b 4.5x^8 dx = 1 \Rightarrow 4.5 \left(\frac{b^9}{9}\right) = 1$$

$$\Rightarrow b^9 = 2 \Rightarrow \boxed{b = 2^{\frac{1}{9}}}$$