1. **(10 pts)** Find the general solution for the following differential equations:

(a) $\frac{dy}{dx} = \frac{x^2+5}{y^3}$.

Separate variables.

$$\int y^3\,dy = \int (x^2+5)\,dx$$

$$\boxed{\frac{y^4}{4} = \frac{x^3}{3} + 5x + C}$$

(b) $yy' - 2xe^{x^2} = 0$.

Separate variables

$$y\frac{dy}{dx} = 2xe^{x^2} \Rightarrow y\,dy = 2xe^{x^2}\,dx$$

Integrating both sides

$$\int y\,dy = \int 2xe^{x^2}\,dx$$

$$\boxed{\frac{y^2}{2} = e^{x^2} + C}$$

2. **(10 pts)** Find the particular solution of the differential equation

$$xy' + y = 5x^4 + 2x,$$

with the initial conditions that when $x = 1, y = 2$.

Rewrite

$$y' + \frac{y}{x} = \frac{5x^4 + 2x}{x}$$

Use First Order D.E. Formula.

$$P(x) = \frac{1}{x} \qquad Q(x) = \frac{5x^4 + 2x}{x}$$

1. Integrating factor $= e^{\int P(x)dx} = e^{\int \frac{1}{x}dx}$

$$= e^{\ln x} = x$$

2. Compute $\int Q(x)\, x\, dx = \int \left(\frac{5x^4 + 2x}{x} \cdot x\right) dx$

$$= \int (5x^4 + 2x)\, dx$$

$$= x^5 + x^2 + C$$

So general solution $y = \frac{1}{x}(x^5 + x^2 + C)$

Use $x = 1$, $y = 2$ $\Rightarrow$ $2 = 1(1 + 1 + C)$

$\Rightarrow C = 0$

So particular solution: $\boxed{y = \frac{1}{x}(x^5 + x^2) = x^4 + x}$

3. **(15 pts)** A tank contains 500 L of pure water. A solution that contains 0.1 kg of sugar per liter enters tank at the rate 10 L/min. The solution is mixed and drains from the tank at the same rate of 10 L/min.

(a) Set up and solve a differential equation for the quantity Q (in kg) of sugar in the tank as a function of time t. Your function should not have any generic constants.

$$\frac{dQ}{dt} = 1 - \frac{10Q}{500} = 1 - \frac{Q}{50}$$

$$\therefore \frac{dQ}{dt} = \frac{50-Q}{50} \Rightarrow \int \frac{dQ}{50-Q} = \int \frac{dt}{50}$$

$$\Rightarrow -\ln|50-Q| = t/50 + C$$

$$\Rightarrow 50 - Q = Ce^{-t/50}$$

At $t=0$, $Q=0$

$$\Rightarrow 50 - 0 = Ce^{-0/50} \Rightarrow C = 50$$

final answer

$$Q = \boxed{50 - 50e^{-t/50}}$$

(b) Find the amount of sugar after 45 minutes.

Plug into solution from part (a).

$$Q = 50 - 50e^{-45/50}$$

$$= \boxed{50 - \frac{50}{e^{45/50}}}$$

You can leave your answer like this!

4. **(15 pts)** Recall Newton's law of cooling which says that the temperature T of a body placed in a surrounding at a temperature T_0 follows the differential equation $\frac{dT}{dt} = k(T - T_0)$. A thermometer is taken from a room where the temperature is $23\,°C$ to the outdoors, where the temperature is $-13\,°C$. After one minute the thermometer reads $7\,°C$. What will the reading on the thermometer be after 3 minutes ?

The solution to the diff. eq. is

$$T = T_0 + Ce^{kt}$$

Need to find C, k. $T_0 = -13$

At $t = 0$, $T = 23$

So $23 = -13 + Ce^{k(0)}$

$\Rightarrow 36 = C$

At $t = 1$, $T = 7$

So $7 = -13 + 36e^{k(1)}$

$\Rightarrow \frac{20}{36} = e^k \Rightarrow k = \ln\left(\frac{20}{36}\right)$

After 3 minutes,

$$\boxed{T = -13 + 36e^{\ln\left(\frac{20}{36}\right)\cdot 3}}$$

You can leave your answer like this.

5. **(10 pts)**

(a) Put the following equation for a sphere in standard form

$$x^2 + y^2 + z^2 - 4y + 6z + 4 = 0.$$

$$(x - 2)^2 + y^2 + (z+3)^2 = 9$$

(b) What is the center and radius of the sphere in 5(a) ?

Center = $(2, 0, -3)$

Radius = 3

6. **(10 pts)** Consider the equation of a quadric surface $x^2 - z^2 = 4y$.

(a) What type of a quadric surface is it ?

Hyperbolic Paraboloid.

(b) Write the equation of the yz-trace of this surface, i.e., the equation for the points lying on the intersection of the yz-plane and the surface.

yz-trace means set $x = 0$.

We get : $0^2 - z^2 = 4y$

$$\Rightarrow \boxed{-z^2 = 4y}$$

7. **(10 pts)** Let $f(x,y) = xe^{(x+y)}$. Compute the following partial derivatives:

(a) $\frac{\partial f}{\partial x} =$ [Think of y as a constant]

$$\frac{\partial}{\partial x}\left(x e^{(x+y)}\right) = e^{(x+y)}\frac{\partial (x)}{\partial x} + x\,\frac{\partial}{\partial x}\left(e^{(x+y)}\right)$$

$$= e^{(x+y)} + x\, e^{(x+y)} \cdot 1$$

$$= \boxed{e^{(x+y)}(x+1)}$$

(b) $\frac{\partial f}{\partial y} =$ [Think of x as a constant)

$$\frac{\partial}{\partial y}\left(x e^{(x+y)}\right) = x\,\frac{\partial}{\partial y} e^{(x+y)} =$$

$$= x\, e^{(x+y)} \cdot \frac{\partial}{\partial y}(x+y)$$

$$= x\, e^{(x+y)} \cdot 1$$

$$= \boxed{x\, e^{(x+y)}}$$