HW 10 Solutions MAT 17B

19.

$$\int \frac{x^3 - x^2 + x - 4}{(x^2+1)(x^2+4)}\,dx$$

Use partial fractions :

$$\frac{x^3 - x^2 + x - 4}{(x^2+1)(x^2+4)} = \frac{Ax + B}{x^2+1} + \frac{Cx + D}{x^2+4}$$

$$= \frac{(Ax+B)(x^2+4) + (Cx+D)(x^2+1)}{(x^2+1)(x^2+4)}.$$

$$= \frac{(A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D)}{(x^2+1)(x^2+4)}.$$

$A + C = 1$ $\quad$ $4A + C = 1$

$B + D = -1$ $\quad$ $4B + D = -4$

$\Rightarrow$ $A = 0$, $C = 1$, $B = -1$, $D = 0$.

$$\therefore \int \frac{x^3 - x^2 + x - 4}{(x^2+1)(x^2+4)}\,dx = \int \frac{-1}{x^2+1} + \frac{x}{x^2+4}\,dx$$

$$= -\int \frac{1}{x^2+1}\,dx + \int \frac{x}{x^2+4}\,dx$$

use $u = x^2+4$ to get

~~$\tan^{-1} x$~~

$$\boxed{-\tan^{-1} x + \frac{1}{2}\ln|x^2+4| + C}$$

20. $\displaystyle\int \frac{x^3-3x^2+x-6}{(x^2+2)(x^2+1)}\,dx$

Irreducible quadratic terms in the denominator.

$$\frac{x^3-3x^2+x-6}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{(Ax+B)(x^2+1) + (Cx+D)(x^2+2)}{(x^2+2)(x^2+1)}$$

$$= \frac{(A+C)x^3 + (B+D)x^2 + (A+2C)x + (B+2D)}{(x^2+2)(x^2+1)}$$

$\Rightarrow$ $A+C=1$, $B+D=-3$, $A+2C=1$, $B+2D=-6$

Use these to get $A=1$, $C=0$

Use these to get $B=0$, $D=-3$

$$\int \frac{x^3-3x^2+x-6}{(x^2+2)(x^2+1)}\,dx = \int \left(\frac{x}{x^2+2} + \frac{-3}{x^2+1}\right)dx$$

$$= \underbrace{\int \frac{x}{x^2+2}\,dx}_{\text{Use substitution } u = x^2+2} + \int \frac{-3}{x^2+1}\,dx$$

$$\int \frac{x^3-3x^2+x-6}{(x^2+2)(x^2+1)}\,dx = \boxed{\frac{1}{2}\ln|x^2+2| - 3\tan^{-1}x + C}$$

13. $\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx$

Substitute $u = x-1$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$\therefore \int \frac{1}{(x-1)^2+1}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1} u$

$\therefore \int \frac{1}{x^2-2x+2}\,dx = \boxed{\tan^{-1}(x-1) + C}$

25. $$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx$$

by completing the square.

Use $u = x-2$.

to get:

$$\int \frac{1}{(x-2)^2+9}\,dx = \boxed{\frac{1}{3}\tan^{-1}\frac{x-2}{3} + C}$$

26. $$\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx$$

by completing the square.

Use $u = x+1$. to get:

$$\int \frac{1}{(x+1)^2+4}\,dx = \boxed{\frac{1}{2}\tan^{-1}\frac{x+1}{2} + C}$$

32. $$\int \frac{1}{x^2-x+2}\,dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}\,dx$$ by completing the square.

Use $u = x-\frac{1}{2}$ to get:

$$\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}\,dx = \boxed{\sqrt{\frac{4}{7}}\tan^{-1}\frac{x-\frac{1}{2}}{\sqrt{7/4}} + C}$$

34. $\displaystyle\int \frac{x^3+1}{x^2+3}\,dx$

Improper rational function
Use long division first.

$$\frac{x^3+1}{x^2+3}$$

$$\begin{array}{r|l} & x \\ \hline x^2+3 & x^3+1 \\ & x^3+3x \\ \hline & -3x+1 \end{array}$$

$$\therefore \quad \frac{x^3+1}{x^2+1} = x + \frac{-3x+1}{x^2+3}$$

$$\therefore \quad \int \frac{x^3+1}{x^2+1}\,dx = \int \left(x + \frac{-3x+1}{x^2+3} \right) dx$$

$$= \frac{x^2}{2} + \int \frac{-3x+1}{x^2+3}\,dx$$

$$\int \frac{-3x+1}{x^2+3}\,dx$$

$$= \underbrace{\int \frac{-3x}{x^2+3}\,dx}_{\text{use substitution}} + \underbrace{\int \frac{1}{x^2+3}\,dx}$$

use substitution

$$u = x^2+3$$

$$\Rightarrow \frac{du}{dx} = 2x$$

$$\Rightarrow \frac{1}{2}du = x\,dx$$

$$\therefore \int \frac{-3x}{x^2+3}\,dx = \int \frac{-3}{u}\left(\frac{1}{2}du\right)$$

$$= -\frac{3}{2}\ln|u|$$

$$= -\frac{3}{2}\ln|x^2+3|$$

$$= \int \frac{1}{3\left(\frac{x^2}{3}+1\right)}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx$$

substitute $u = \frac{x}{\sqrt{3}}$

to get

$$\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$\therefore \int \frac{-3x+1}{x^2+3} = \boxed{\frac{3}{2}\ln|x^2+3| + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

$$\therefore \int \frac{x^3+1}{x^2+1}\,dx = \boxed{\frac{x^2}{2} - \frac{3}{2}\ln|x^2+3| + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

39. $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$

Use $u = x^2+1$ to get

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|$$

$$\therefore \int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|\Big|_0^1$$

$$= \frac{1}{2}\ln|1^2+1| - \frac{1}{2}\ln|0^2+1|$$

$$= \frac{1}{2}\ln|2|$$

52. $\displaystyle\int \frac{1}{(x+1)^2(x^2+1)}\,dx$

Repeated linear term AND irreducible quadratic:

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2}{(x+1)^2(x^2+1)}$$

$$= \frac{A(x^3+x^2+x+1)+B(x^2+1)+(Cx+D)(x^2+2x+1)}{(x+1)^2(x^2+1)}$$

$$= \frac{(A + C)x^3 + (A + B + 2C + D)x^2 + (A + C + 2D)x + (A + B + D)}{(x+1)^2(x^2+1)}$$

$\Rightarrow \quad A + C = 0, \quad A + B + 2C + D = 0$

$A + C + 2D = 0, \quad A + B + D = 1$

Use $-A = C$ in the other 3.

$$A + B + 2(-A) + D = 0$$
$$A + (-A) + 2D = 0$$
$$A + B + D = 1$$

Simplifying: $\quad -A + B + D = 0$

$$2D = 0$$
$$A + B + D = 1$$

$D = 0, \quad -A + B = 0, \quad A + B = 1$

$\Rightarrow \quad A = \frac{1}{2}, \; B = \frac{1}{2}$

$\Rightarrow \quad C = -\frac{1}{2}$

$$\therefore \int \frac{1}{(x+1)^2(x^2+1)} = \int \frac{\left(\frac{1}{2}\right)}{x+1} + \frac{\left(\frac{1}{2}\right)}{(x+1)^2} + \frac{-\frac{1}{2}x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{2}\underbrace{\int \frac{x}{x^2+1}\,dx}$$

Use substitution

$u = x^2+1$

to get

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|$$

$$\therefore \int \frac{1}{(x+1)^2(x^2+1)}\,dx = \boxed{\frac{1}{2}\ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\ln|x^2+1| + C}$$