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LONG MONOTONE PATHS IN ABSTRACT POLYTOPES*

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As is now well known, the simplex method, under its various pivoting rules, is not a "good algorithm" in the sense of J. Edmonds, i.e., the number of pivots needed to solve a given linear programming problem by this method cannot be bounded by a polynomial function of the number of rows and columns defining it. Klee, Minty and Jerosolow have developed methods for constructing examples of such linear programs on polytopes. The aim of this paper is to extend these constructions to abstract polytopes.

1. Introduction. In a recent paper, Klee and Minty [5] constructed a special class of linear programming problems and demonstrated that the simplex method (using certain pivot rules) is not a "good algorithm" in the sense of J. Edmonds. By this is meant that the number of pivots required for solving a given linear program cannot be bounded by a polynomial function of the parameters that determine it, namely the number of rows and columns.

They constructed examples requiring a large number of pivots using the usual pivot rules, namely the "random pivot rule" where one moves to any better adjacent vertex, and the "min $\bar{c}_j$ rule," where one moves to the adjacent vertex which gives the best per-unit improvement. In a subsequent paper, Jerosolow [4] has shown that the constructive procedure used by Klee and Minty can be used to demonstrate similar behavior under the "best adjacent vertex" rule as well.

Abstract polytopes provide a convenient framework for investigating the combinatorial structure of simple polytopes (nondegenerate bounded systems of linear inequalities). An abstract polytope is a combinatorial system given by a set of three axioms (proposed by G. B. Dantzig). Many known results for simple polytopes have been shown (by combinatorial arguments) to hold for abstract polytopes. Some references in this area are Adler [1], Adler and Dantzig [2], Adler, Dantzig and Murty [3], Murty [7].

The aim of this paper is to introduce the notion of an objective function on an abstract polytope, and thus produce problems similar to those of Klee and Minty [5] and Jerosolow [4] on these structures. Since polytopes are abstract polytopes; our results are not new. Also, they cannot be readily extended to "geometrical" polytopes without added complexity, and thus, in no way diminish the significance of the methods of [4] and [5].

In §2 of this paper, we introduce the axioms of an abstract polytope, and relate them to those of a simplicial complex and pseudomanifold. In §3 we systematically introduce the notion of an objective function on an abstract polytope, and in §4 we obtain results similar to those of [5] and [4] on these structures.

2. Simplicial Complexes, Pseudomanifolds and Abstract Polytopes. In this section, we relate the concept of an abstract polytope to the well-studied concepts of simplicial complex and pseudomanifold. We shall follow the terminology of Spanier [9].

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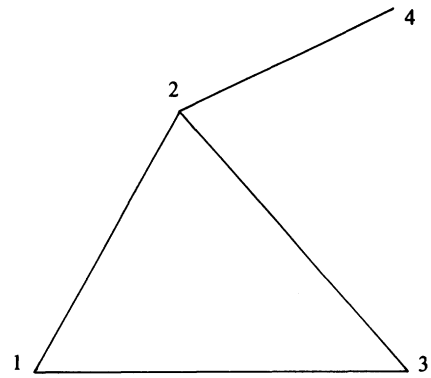

FIGURE 2.1

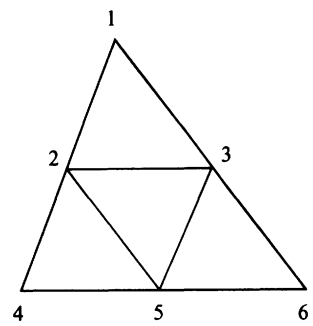

FIGURE 2.2

Given a finite set T of symbols, called vertices, and a set V of finite nonempty subsets of T, called simplexes, $P = (T, V)$ constitutes a simplicial complex if:

(2.1) Any set consisting of exactly one vertex is a simplex.

(2.2) Any nonempty subset of a simplex is a simplex.

Any simplex consisting of $q + 1$ vertices is called a q dimensional simplex, or q-simplex for short, and any subset $v' \subset v$ of $r + 1$ vertices is called an r-face of v.

As we shall see subsequently, abstract polytopes are also simplicial complexes. We now give an example of a simplicial complex: $T = \{1, 2, 3, 4\}$ and the simplexes V are the subsets $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{2, 4\}$ $\{1, 2, 3\}$. See Figure 2.1 as well.

Another related concept is that of a pseudomanifold. Given a simplicial complex $P = (T, V)$, we call it a pseudomanifold of dimension d if

(2.3) Every simplex is a face of some d-simplex of P.

(2.4) Every $(d - 1)$-simplex is a face of at most two d-simplexes of P.

(2.5) If v and v' are d-simplexes, there is a finite sequence $v = v_1, v_2, \ldots, v_m = v'$ of d simplexes of P such that v_i and v_{i+1} have a $(d - 1)$-face in common, for $1 \leqslant i < m$.

The boundary of a d-dimensional pseudomanifold P, denoted by ∂P, is defined to be the subcomplex of P generated by $(d - 1)$-simplexes which are faces of exactly one d-simplex of P. As we shall subsequently see, abstract polytopes are pseudomanifolds

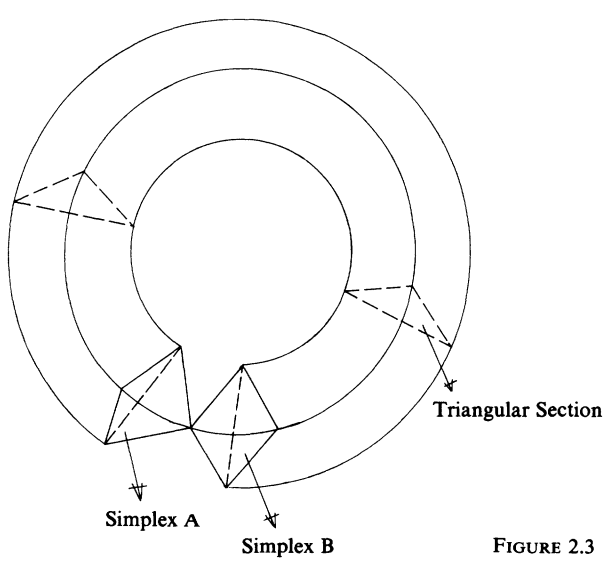

FIGURE 2.3

without boundary. Pseudomanifolds have been studied in relation to mathematical programming structures; see for example Saigal [8], Lemke and Grotzinger [6].

As an example of a pseudomanifold $P = (T, V)$, consider $T = \{1, 2, 3, 4, 5, 6\}$ and the 2-simplexes $\{1, 2, 3\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 5, 6\}$. This is a pseudomanifold with a boundary, the boundary being the subsimplicial complex $P' = \{T, V'\}$ $T = \{1, 2, 3, 4, 5, 6\}$, where the simplexes V' are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{1, 2\}$, $\{2, 4\}$, $\{4, 5\}$, $\{5, 6\}$, $\{3, 6\}$, $\{1, 3\}$. It can be readily confirmed that P' is a pseudomanifold without boundary.

We now introduce the concept of an abstract polytope in this setting. Given a $(d-1)$-dimensional pseudomanifold $P = (T, V)$, we call it a d-dimensional abstract polytope if:

(2.6) it has no boundary;

(2.7) in the sequence of axiom (2.5) we require, in addition, that $v \cap v' \subset v_i$ for each $i = 1, \ldots, m$.

Any triangulation (including the six triangles shown) of surface of the object in Figure 2.3 is a two dimensional pseudomanifold without boundary and violates the axiom (2.7) for the simplexes marked A and B.

We now relate this development to that of Dantzig. To do so, consider a graph whose vertices are the $(d-1)$-simplexes of the abstract polytopes. An edge connects two vertices (in the graph) if and only if these $(d-1)$ simplexes share a common $(d-2)$ dimensional face. (From the axioms, this graph is well defined.) In the standard development of an abstract polytope [1], $(d-1)$ simplexes are referred to as "vertices," and thus the system given by the following axioms can be readily seen equivalent to one developed above:

(2.8) Every vertex of P has cardinality d.

(2.9) Any subset of $(d-1)$-elements of T is either contained in no vertices of P or in exactly two (called neighbors or adjacent vertices).

(2.10) Given any pair of vertices v, v' in V, there exists a sequence of vertices $v = v_1, v_2, \ldots, v_m = v'$ such that

(a) v_i, v_{i+1} are neighbors, $i = 1, \ldots, m-1$,

(b) $v \cap v' \subset v_i$, $i = 1, \ldots, m$.

To be consistent with other works in the area and as axioms (2.8)–(2.10) are more suggestive of the connection with linear programming, we will use these to define an abstract polytope in the subsequent developments. We will also assume that $T = \bigcup \{v : v \in V\}$.

Given an abstract polytope $P = (T, V)$ and an arbitrary set $U \subset T$ with $|U| = k \leqslant d$, if $F(P \mid U) = \{v \in V \mid U \subset v\}$ is nonempty, we say $F(P \mid U)$ is a face of P. It can be easily shown that $\{v \backslash U \mid v \in F(P \mid U)\}$ is a $(d-k)$-dimensional abstract polytope, and thus we call $F(P \mid U)$ a $(d-1)$-dimensional face of P. We define the 0, 1 and $(d-1)$-dimensional faces of P as vertices, edges and facets respectively. Hence, if $|T| = n$, P has n-facets.

Let $P(d, n)$ be the class of all d-dimensional abstract polytopes with n-facets. In addition, for the ease of exposition, we shall misuse notation to represent $P \in P(d, n)$ as an abstract polytope as well as its vertices whenever there is no chance of confusion.

Following Adler [1], given $P \in P(d_1, n_1)$ and $Q \in P(d_2, n_2)$, we define $P \otimes Q \in P(d_1 + d_2, n_1 + n_2)$ as the abstract polytope $P \otimes Q = \{(u, v) \mid u \in P, v \in Q\}$, where $(u, v) = u \cup v$. Also, F is a face of $P \otimes Q$ if there are faces F_P of P and F_Q of Q such that $F = F_P \otimes F_Q$.

3. Objective Functions on Abstract Polytopes. We are now ready to define an objective function on an abstract polytope.

Given a d-dimensional abstract polytope P, we define a sequence of distinct vertices $v_0, v_1, \ldots, v_l$ of P as a path of length l from v_0 to v_l if the vertices v_i and v_{i+1} are adjacent for each $i = 0, \ldots, l-1$. In addition, for ease of notation, by $p_l(v, v^1)$ or $q_l(v, v^1)$ we shall represent a specific path of length l between v and v^1, and by $p(v, v^1)$ or $q(v, v^1)$ a specific path of some length between v and v^1. We shall drop the subscript l on p or q whenever the length is clear from the context.

Given a real valued one-to-one map $\phi : P \to R$, and a face F of P, we define:

(3.1) $\bar{v} \in F$ as a ϕ-max vertex if $\phi(\bar{v}) \geqslant \phi(v)$ for all $v \in F$,

(3.2) $\underline{v} \in F$ as a ϕ-min vertex if $\phi(\underline{v}) \leqslant \phi(v)$ for all $v \in F$,

(3.3) $p_l(v_0, v_l)$ as a ϕ-increasing path of length l if $\phi(v_0) < \phi(v_1) < \cdots < \phi(v_l)$; a ϕ-decreasing path of length l if $\phi(v_0) > \phi(v_1) > \cdots > \phi(v_l)$; a *strict* ϕ-increasing path of length l if it is a ϕ-increasing path and $\phi(v_{i+1}) \geqslant \phi(v)$ for all $v \in N(v_i)$, where $N(v_i)$ are the vertices adjacent to v_i, $i = 1, 2, \ldots, l-1$; and a strict ϕ-decreasing path of length l if it is ϕ-decreasing and $\phi(v_{i+1}) \leqslant \phi(v)$ for all $v \in N(v_i)$, $i = 1, 2, \ldots, l-1$.

(3.4) ϕ-as an *objective function on* P if for each face F of P, and each $v \in F$ there exists a ϕ-increasing path $p(v, \bar{v})$ in F and a ϕ-decreasing path $p(v, \underline{v})$ in F where $\bar{v}$ and $\underline{v}$ are respectively the ϕ-max and ϕ-min vertices in F. Let $\Phi(P)$ be the set of all objective functions on P.

We say that an abstract polytope $P \in P(d, n)$ is *reversible* of length l if there is a ϕ in $\Phi(P)$ and a pair of vertices v_0 and v_l for which there exists a strict ϕ-increasing path of length l $p_l(v_0, v_l) = v_0, v_1, \ldots, v_l$ and a strict ϕ-decreasing path of length l $q_l(v_l, v_0) = v_l(= w_0), w_1, \ldots, w_l(= v_0)$. Given an objective function ϕ in $\Phi(P)$, following Klee and Minty [5], we define the *ϕ-height* of P as the maximum of lengths of the various ϕ-increasing paths in P, and the *height* of P as the maximum ϕ-height of over all ϕ in $\Phi(P)$. Also, we define the *strict ϕ-height* of a reversible polytope $P \in P(d, n)$ as its maximal reversible length and the strict height as the maximum strict ϕ-height as ϕ ranges over all of $\Phi(P)$. Now, by $H_a(d, n)$ we represent the maximum height over all P in $P(d, n)$, and $M_a(d, n)$ as the maximum strict height as P ranges over $P(d, n)$.

Given a path $p(v_0, v_l)$ in a face F of some abstract polytope P, and a vertex u of some abstract polytope Q, we define $u \otimes p(v_0, v_l)$ as the path $(u, v_0), (u, v_1), \ldots, (u, v_l)$ in the face $\{u\} \otimes F$ of $Q \otimes P$; and $p(v_0, v_l) \otimes u$ as the path $(v_0, u), (v_1, u), \ldots, (v_l, u)$ in the face $F \otimes \{u\}$ of the abstract polytope $P \otimes Q$.

We now prove a lemma which establishes a result on objective functions on abstract polytopes.

LEMMA. *Let $P \in P(d, n)$, $\phi \in \Phi(P)$ with $0 \leqslant \phi(v) \leqslant 1$ for all $v \in P$, and $Q = \{u_1, u_2, \ldots, u_k\} \in P(2, k)$ (where u_i, u_{i+1} are adjacent vectices of Q). If $f(u_i)$, $g(u_i)$, $i = 1, \ldots, k$, are two strict monotone sequences of real numbers with $g(u_i) \neq f(u_j)$ for all i, j then $\psi(u, v) = (1 - \phi(v))f(u) + \phi(v)g(u)$ is in $\Phi(Q \otimes P)$.*

PROOF. Let F be a face of $Q \otimes P$. From [1], $F = F_Q \otimes F_P$, where F_Q and F_P are faces of Q and P respectively. Let $\bar{v}$, $\underline{v}$, $p(v, \bar{v})$, $p(v, \underline{v})$ be the ϕ-max, the ϕ-min, a ϕ-increasing path and a ϕ-decreasing path respectively in F_P. These paths exist since $\phi \in \Phi(P)$. Also, define

$$\begin{aligned} \bar{w}_i &= \bar{v} \quad \text{if } g(u_i) > f(u_i), \\ &= \underline{v} \quad \text{if not.} \\ \underline{w}_i &= \underline{v} \quad \text{if } g(u_i) > f(u_i), \\ &= \bar{v} \quad \text{if not.} \end{aligned}$$

We now consider the three cases depending on whether F_Q is a vertex, an edge or the whole polytope Q.

Case (i). F_Q is a vertex. Let $F_Q = \{u_i\}$. It is easily verified that $(u_i, \bar{w}_i)$ and $(u_i, \underline{w}_i)$ are respectively the ϕ-max and ϕ-min vertices in F. Also, from an arbitrary vertex (u_i, v) in F, $u_i \otimes p(v, \bar{w}_i)$ and $u_i \otimes p(v, \underline{w}_i)$ are the ψ-increasing and ψ-decreasing paths in F.

Case (ii). F_Q is an edge. Let $F_Q = \{u_i, u_{i+1}\}$. Then $(u_i, \underline{w}_i)$ and $(u_{i+1}, \bar{w}_{i+1})$ are respectively the ϕ-min and ϕ-max vertices of F. Also, from an arbitrary vertex (u_i, v) in F, (u_i, v), (u_{i+1}, v), $u_{i+1} \otimes p(v, \bar{w}_{i+1})$ and $u_i \otimes p(v, \underline{w}_i)$ are respectively the ψ-increasing and ψ-decreasing paths in F. Similarly, one can construct the required paths from an arbitrary vertex (u_{i+1}, v).

Case (iii). $F_Q = Q$. Then $(u_1, \underline{w}_i)$ and $(u_k, \bar{w}_k)$ are the ϕ-min and ϕ-max vertices respectively in F. Let (u_i, v) be an arbitrary vertex in F. Define $p(u_i, u_k) = u_i, u_{i+1}, \ldots, u_k$, and $p(u_i, u_1) = u_i, u_{i-1}, \ldots, u_1$. Then $p(u_i, u_k) \otimes v$, $u_k \otimes p(v, \bar{w}_k)$ and $p(u_i, u_1) \otimes v$, $u_1 \otimes p(v, \underline{w}_1)$ are respectively the ϕ-increasing and ϕ-decreasing paths in F.

4. Long Monotone Paths in Abstract Polytopes. In this section, we display a special class of abstract polytopes for which there exist "long" ϕ-increasing paths and strict ϕ-increasing paths. We recall that $H_a(d, n)$ bounds the length of a ϕ-increasing path on any P in $P(d, n)$ and $M_a(d, n)$ bounds the length of a strict ϕ-increasing path on any reversible polytope P in $P(d, n)$. In analogy with linear programming $H_a(d, n)$ represents the maximal number of pivots required to solve a problem when the "random pivot rule" is used, $M_a(d, n)$ represents the number of pivots required when the "best adjacent vertex" pivot rule is used.

We now establish a result on $H_a(d, n)$.

THEOREM 1. $H_a(d+2, n+k) \geqslant kH_a(d, n) + k - 1$.

PROOF. Let $P \in P(d, n)$ such that there is a $\phi \in \Phi(P)$ and a ϕ-increasing path $p(v_0, v_l) = v_0, v_1, \ldots, v_l$, of length $l = H_a(d, n)$. Let $Q = \{u_1, u_2, \ldots, u_k\} \in P(2, k)$, with u_i and u_{i+1} neighbors. Define $f(u_i)$, $g(u_i)$, $i = 1, \ldots, k$, as two strictly monotone sequences of distinct real numbers, such that $g(u_i) - f(u_i) = (-1)^i$, $i = 1, \ldots, k$. Also, assume without loss of generality, that $0 \leqslant \phi(v) \leqslant 1$ for all $v \in P$. Hence, from Lemma, $\psi(u, v) = (1 - \phi(v))f(u) + \phi(v)g(u)$ is in $\Phi(Q \otimes P)$.

Now $p(v_l, v_0) = v_l, v_{l-1}, \ldots, v_0$ is a ϕ-decreasing path. Then, for k-odd,

$$u_1 \otimes p(v_0, v_l), \quad u_2 \otimes p(v_l, v_0), \ldots, u_k \otimes p(v_0, v_l)$$

and for k-even

$$u_1 \otimes p(v_0, v_l), \quad u_2 \otimes p(v_l, v_0), \ldots, u_k \otimes p(v_l, v_0)$$

is a ψ-increasing path in $Q \times P$ of length $kl + k - 1$. Hence, the result.

The following theorem establishes a similar result for $M_a(d, n)$.

THEOREM 2. $M_a(d+2, n+4k+1) \geqslant 2kM_a(d, n) + 4k - 2$.

PROOF. Let $P \in P(d, n)$ and be reversible of length $l = M_a(d, n)$; $Q = \{u_1, u_2, \ldots, u_{4k+1}\} \in P(2, 4k+1)$ with u_i, u_{i+1} as neighbors; $\phi \in \Phi(P)$ which achieves the reversible length l with $0 \leqslant \phi(v) \leqslant 1$ with $\phi(\underline{v}) = 0$, $\phi(\bar{v}) = 1$ where $\underline{v}$ and $\bar{v}$ are respectively the ϕ-min and ϕ-max vertices in P. Let $p(v_0, v_l) = v_0, v_1, \ldots, v_l$ and $q(v_l, v_0) = v_l (= w_0), w_1, \ldots, w_l (= v_0)$ be the strict ϕ-increasing and strict ϕ-decreasing paths of length $l = M_a(d, n)$ respectively in P.

Define $\theta > 0$, $\delta > 0$ so that $\theta < \phi(v_{i+1}) - \phi(v_i)$ for $i = 0, \ldots, l$ and $\theta < \phi(w_i) - \phi(w_{i+1})$, $i = 0, \ldots, l$. Also, assume that $\phi(v_0) < \theta$, $1 - \phi(v_l) < \theta$. Define $\delta > 0$ such that $\delta < \theta - \phi(v_0)$.

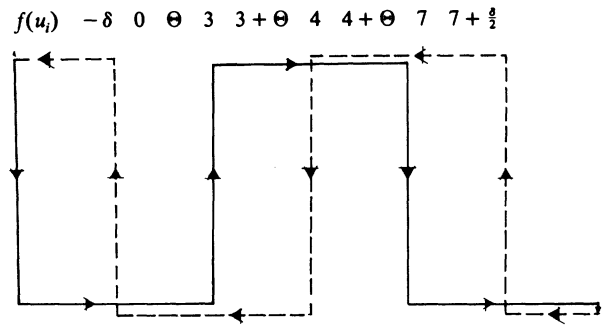

FIGURE 4.1

Assign the following pairs $f(u_i)$, $g(u_i)$, of real numbers to the vertices of Q : $f(u_1) = -\delta$, $f(u_{2i})$, $i = 1, 2, \ldots, 2k$, be the sequence $0, 3, 4, 7, 8, 11, \ldots, f(u_{2i+1}) = f(u_{2i}) + \theta$, $i = 1, \ldots, 2(k-1)$, $f(u_{4k+1}) = f(u_{4k}) + \delta/2$; and $g(u_1) = 1 - \delta$, $g(u_{2i})$, $i = 1, \ldots, 2(k-1)$, be the sequence $1, 2, 5, 6, 9, 10, \ldots,$ with $g(u_{4k}) = f(u_{4k}) + \delta$, $g(u_{2i+1}) = g(u_{2i}) + \theta$, $i = 1, \ldots, 2(k-1)$, and $g(u_{4k+1}) = g(u_{4k}) + \delta$.

It can be readily verified that f, g and u satisfy the conditions of the Lemma, and thus $\psi(u, v) = (1 - \phi(v))f(u) + \phi(v)g(u)$ is in $\Phi(Q \otimes P)$. We now demonstrate that $Q \otimes P$ is a reversible polytope of length $2kl + 4k - 2$ in $P(d+2, n+4k+1)$, from which the result follows.

It can be readily checked (see Figure 4.1) that the strict ϕ-increasing path from (u_2, v_0) to (u_{4k}, v_l) is

$$u_2 \otimes p(v_0, v_l), (u_3, v_l), u_4 \otimes q(v_l, v_0), (u_5, v_0), \ldots, u_{4k-2} \otimes p(v_0, v_l),$$

$$(u_{4k-1}, v_l), (u_{4k}, v_l)$$

and its length is $2kl + 4k - 2$.

Also, the strict ϕ-decreasing path from (u_{4k}, v_l) to (u_2, v_0) is (u_{4k}, v_l), $u_{4k-1} \otimes q(v_l, v_0)$, (u_{4k-2}, v_0), $u_{4k-3} \otimes p(v_0, v_l), \ldots, u_3 \otimes q(v_l, v_0)$, (u_2, v_0), and has length $2kl + 4k - 2$.

Now, $\underline{w} = (u_1, \underline{v})$ and $\overline{w} = (u_{4k+1}, \overline{v})$ are the ψ-min and ψ-max vertices in $Q \times P$. Note that

$$\bar{\psi}(w) = (\psi(w) - \psi(\underline{w}))/(\psi(\overline{w}) - \psi(w)) \in \Phi(Q \times P)$$

and that there is a $\theta > 0$ and $\delta > 0$ as was required for the $\phi \in \Phi(P)$.

Thus, the result follows.

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