Practice Midterm II
MAT-25    Advanced Calculus
Spring 2013

Name ________________________________

• This test is closed notes, closed book.

• You may use any definition/theorem/property discussed from class without proof - if you
do, you must write the theorem statement in full detail.

• There are 10 pages and 7 questions total.

• The maximum score in the test is 85 points.

• IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY,
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Signature ____________________________________________
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Properties of absolute value function:

1. $|x| \geq 0 \quad \forall x \in \mathbb{R}$

2. $|x| < r$ if and only if $-r < x < r \quad \forall x, r \in \mathbb{R}$

3. $|-x| = |x| \quad \forall x \in \mathbb{R}$

4. $-|x| \leq x \leq |x| \quad \forall x \in \mathbb{R}$

5. $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$

6. $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$

7. $|x| - |y| \leq |x - y| \quad \forall x, y \in \mathbb{R}$ and $|y| - |x| \leq |x - y| \quad \forall x, y \in \mathbb{R}$

You can use the following fact without proof:

\[ |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases} \]
1. (a) **(5 pts)** Give the precise definition of the limit of a sequence \( \{s_n\}_{n=1}^{\infty} \).

(b) **(5 pts)** Explain the meaning of the sentence “\( E \subseteq \mathbb{R} \) is dense in \( \mathbb{R} \)” in mathematical terms.
2. **(10 pts)** Let $f(x) = \frac{1}{x^2}$. Find the infimum of the set $E = \{f(x) : x \in \mathbb{R}\}$ and justify your answer.
3. (15 pts) Let $E \subseteq \mathbb{R}$ be any nonempty subset of real numbers that is bounded below. Let $L = \inf E$. Show that there exists a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} s_n = L$. 
4. (15 pts) Let \( r \in \mathbb{R} \) be a fixed real number. Let \( A \subseteq \mathbb{R} \) be a subset of real numbers. Let \( B = \{ x - r : x \in A \} \). Find a relation between \( \sup(A) \) and \( \sup(B) \), and prove it.
5. (15 pts) Let $E$ be a subset of real numbers of the form
\[ E = \left\{ \frac{\sqrt{2}m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}. \]
Show that $E$ is dense in $\mathbb{R}$. [Hint: Consider any interval $(a, b)$. Manipulate this interval and use density of $\mathbb{Q}$.]
6. (10 pts) Find

$$\lim_{n \to \infty} \frac{2n^2 + 1}{3n^2 + 2}$$

and justify your answer.
7. (10 pts) Show that \( \min\{x, y\} = \frac{(x+y)}{2} - \frac{|x-y|}{2} \).