1. (a) (5 pts) Give the precise definition of the limit of a sequence \(\{s_n\}_{n=1}^{\infty}\).

We say \( \lim_{n \to \infty} s_n = L \) if

\[ \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} \ (n \geq N) \Rightarrow (|s_n - L| < \varepsilon). \]

Also acceptable:

\[ \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \ |s_n - L| < \varepsilon. \]

(b)(5 pts) Explain the meaning of the sentence "\(E \subseteq \mathbb{R}\) is dense in \(\mathbb{R}\)" in mathematical terms.

\(E\) is dense in \(\mathbb{R}\) means, for every interval \((a, b)\) with \(a < b\), \(\exists y \in E\) such that \(y \in (a, b)\).
2. (10 pts) Let \( f(x) = \frac{1}{x^2} \). Find the infimum of the set \( E = \{ f(x) : x \in \mathbb{R} \} \) and justify your answer.

\[
\inf E = 0.
\]

We use the theorem that if \( L \) is a lower bound for \( E \) then

\[
L = \inf E \iff \forall \epsilon > 0, \exists y \in E \text{ s.t. } y < L + \epsilon.
\]

Since \( f(x) = \frac{1}{x^2} \geq 0 \) \( \forall x \in \mathbb{R} \), since every square is nonnegative.

0 is a lower bound for \( E \).

Consider any \( \epsilon > 0 \).

Let \( x = \sqrt{\frac{2}{\epsilon}} \)

Let \( y = f(x) = \frac{1}{x^2} = \frac{\epsilon}{2} \).

\( y \in E \)

Also, \( y = \frac{\epsilon}{2} < \epsilon = 0 + \epsilon \).
3. **(15 pts)** Let $E \subseteq \mathbb{R}$ be any nonempty subset of real numbers that is bounded below. Let $L = \inf E$. Show that there exists a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} s_n = L$.

We use the theorem that $L = \inf E \Rightarrow \forall \varepsilon > 0$, $\exists y \in E$ s.t. $y < L + \varepsilon$.

We construct a sequence as follows:

Given $n \in \mathbb{N}$, we choose $y \in E$ s.t. $y < L + \frac{1}{n}$.

Let $s_n = y$. (The $y$ depends on $n$)

Clearly $s_n \in E$.

**Claim:** $\lim_{n \to \infty} s_n = L$.

Need to show: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $|s_n - L| < \varepsilon$ \(\forall n \geq N\).

1. Consider any $\varepsilon > 0$.
2. Let $N$ be a natural number larger than $\frac{1}{\varepsilon}$ (this exists by the Archimedean principle).
3. Consider any $n > N$.
4. We know by construction $s_n < L + \frac{1}{n} \leq L + \frac{1}{N} \leq L + \varepsilon$.

   $\Rightarrow s_n - L < \varepsilon$  
   $\Rightarrow |s_n - L| < \varepsilon$.  


4. (15 pts) Let \( r \in \mathbb{R} \) be a fixed real number. Let \( A \subseteq \mathbb{R} \) be a subset of real numbers. Let \( B = \{x - r : x \in A\} \). Find a relation between \( \text{sup}(A) \) and \( \text{sup}(B) \), and prove it.

**Claim:** \( \text{sup}(A) - r = \text{sup}(B) \).

**Proof:**

1. We first prove \( \text{sup}(A) - r \geq \text{sup}(B) \).
   
   We know \( \text{sup}(A) \geq x \quad \forall x \in A \) \quad [\text{def. of sup}(A)]

   \[ \Rightarrow \text{sup}(A) - r \geq x - r \quad \forall x \in A. \]

   \[ \Rightarrow \text{sup}(A) - r \geq y \quad \forall y \in B. \]

   \[ \Rightarrow \text{sup}(A) - r \geq \text{sup}(B) \quad [\text{def. of sup}(B)] \]

2. We next prove \( \text{sup}(A) - r \leq \text{sup}(B) \).

   We know \( x \leq \text{sup}(B) \) \quad \forall x \in B \quad [\text{def. of sup}(B)]

   \[ \Rightarrow x + r \leq \text{sup}(B) + r \quad \forall x \in B \]

   \[ \Rightarrow y < \text{sup}(B) + r \quad \forall y \in A \]

   \[ \Rightarrow \text{sup}(A) \leq \text{sup}(B) + r \quad [\text{def. of sup}(A)] \]

   \[ \therefore \text{sup}(A) - r = \text{sup}(B). \]
5. (15 pts) Let $E$ be a subset of real numbers of the form

$$E = \{ \frac{\sqrt{2m}}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \}.$$ 

Show that $E$ is dense in $\mathbb{R}$. [Hint: Consider any interval $(a, b)$. Manipulate this interval and use density of $\mathbb{Q}$]

Consider any $(a, b)$ with $a < b$

Let consider the interval $\left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$,

\[ \exists \gamma \in \mathbb{Q} \text{ st. } \gamma \in \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right). \]

Also since $\gamma \in \mathbb{Q}$ , \( \exists m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \text{ st.} \)

\[ \gamma = \frac{m}{n}. \]

\[ \Rightarrow \frac{m}{n} \in \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right) \Rightarrow \frac{a}{\sqrt{2}} < \frac{m}{n} < \frac{b}{\sqrt{2}} \]

\[ \Rightarrow a < \frac{\sqrt{2}m}{n} < b. \]

\[ \Rightarrow \frac{\sqrt{2}m}{n} \in (a, b). \]

Since $\frac{\sqrt{2}m}{n} \in E$. Let let $x = \frac{\sqrt{2}m}{n}$.

and $x \in E \text{ and } x \in (a, b)$. \( \square \)
6. (10 pts) Find

\[
\lim_{n \to \infty} \frac{2n^2 + 1}{3n^2 + 2}
\]

and justify your answer.

**Claim:** \[\lim_{n \to \infty} \frac{2n^2 + 1}{3n^2 + 2} = \frac{2}{3}\]

**Proof:**

1. Consider any \( \epsilon > 0 \).
2. Let \( N \) be any natural number larger than \( \frac{1}{3\sqrt{e}} \) (by Archimedes' principle).
3. Consider \( n \geq N \).

\[
|s_n - L| = \left| \frac{2n^2 + 1}{3n^2 + 2} - \frac{2}{3} \right| = \left| -\frac{1}{3(3n^2 + 2)} \right| = \frac{1}{3(3n^2 + 2)}
\]

Since \( n \geq N \) \( \Rightarrow n^2 \geq N^2 \geq \frac{1}{9\epsilon} \)

\[
\Rightarrow 3n^2 \geq \frac{1}{3\epsilon} \Rightarrow 3n^2 + 2 \geq \frac{1}{3\epsilon} \Rightarrow \epsilon > \frac{1}{3(3n^2 + 2)}
\]

\[
\therefore |s_n - L| = \frac{1}{3(3n^2 + 2)} < \epsilon
\]

Choose \( N \) such that \( \frac{1}{3\sqrt{e}} < \frac{1}{3(3n^2 + 2)} \).
7. (10 pts) Show that \( \min\{x, y\} = \frac{(x+y)}{2} - \frac{|x-y|}{2} \).

We proceed by considering 2 cases:

Case 1: \( x \geq y \).
   
   \[ \Rightarrow x-y \geq 0 \Rightarrow |x-y| = x-y. \]
   
   \[ \Rightarrow \frac{x+y}{2} - \frac{|x-y|}{2} = \frac{x+y}{2} - \frac{x-y}{2} = \frac{x}{2} \]

Also \( \min \{x, y\} = y \) since \( x \geq y \).

Thus \( \min \{x, y\} = y = \frac{x+y}{2} - \frac{|x-y|}{2} \).

Case 2: \( x < y \).
   
   \[ \Rightarrow x-y < 0 \Rightarrow |x-y| = y-x. \]
   
   \[ \Rightarrow \frac{x+y}{2} - \frac{|x-y|}{2} = \frac{x+y}{2} - \frac{y-x}{2} = \frac{x}{2} \]

Also \( \min \{x, y\} = x \) since \( x < y \).

Thus \( \min \{x, y\} = x = \frac{x+y}{2} - \frac{|x-y|}{2} \).

In both cases \( \min \{x, y\} = \frac{x+y}{2} - \frac{|x-y|}{2} \). \( \Box \).