1. (a) **(5 pts)** Give the precise definition of the limit of a sequence $\{s_n\}_{n=1}^{\infty}$.

We say $\lim_{n\to\infty} s_n = L$ if

$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\ (n \geq N) \Rightarrow (|s_n - L| < \epsilon)$.

Also acceptable:

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$,
$|s_n - L| < \epsilon$.

(b) **(5 pts)** Explain the meaning of the sentence "$E \subseteq \mathbb{R}$ is dense in $\mathbb{R}$" in mathematical terms.

E is dense in $\mathbb{R}$ means for every interval (a,b) with $a < b$, $\exists y \in E$ such that $y \in (a,b)$

2. **(10 pts)** Let $f(x) = \frac{1}{x^2}$. Find the infimum of the set $E = \{f(x) : x \in \mathbb{R}\}$ and justify your answer.

$\inf E = 0.$

We use the theorem that if L is a lower bound for E then

$L = \inf E \iff \forall \epsilon > 0, \exists y \in E$ s.t. $y < L + \epsilon$.

Since $f(x) = \frac{1}{x^2} \geq 0 \; \forall x \in \mathbb{R}$, since every square is nonnegative.

0 is a lower bound for E.

Consider any $\epsilon > 0$.

Let $x = \sqrt{\frac{2}{\epsilon}}$

Let $y = f(x) = \frac{1}{x^2} = \epsilon/2$.

$y \in E$

Also. $y = \epsilon/2 < \epsilon. = 0 + \epsilon.$

3. **(15 pts)** Let $E \subseteq \mathbb{R}$ be any nonempty subset of real numbers that is bounded below. Let $L = \inf E$. Show that there exists a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = L$.

We use the theorem that

$L = \inf E \Rightarrow \forall \epsilon > 0, \exists y \in E$ s.t. $y < L + \epsilon$.

We construct a sequence as follows:

Given $n \in \mathbb{N}$, ~~define $s_n \in E$~~ $\exists y \in E$ s.t. $y < L + \frac{1}{n}$.

Let $S_n = y$. (the y depends on n of course)

Clearly $S_n \in E$.

Claim: $\lim_{n\to\infty} S_n = L$.

Need to show: $\forall \epsilon > 0, \exists N \in \mathbb{N}, |s_n - L| < \epsilon \ \forall n \geq N$.

1. Consider any $\epsilon > 0$.
2. Let N be a natural number larger than $\frac{1}{\epsilon}$ (this exists by the Archimedean principle).
3. Consider any $n \geq N$.
4. We know by construction $S_n < L + \frac{1}{n} \leq L + \frac{1}{N} \leq L + \epsilon$

$\Rightarrow S_n - L < \epsilon$

$\Rightarrow |S_n - L| < \epsilon$.

4. **(15 pts)** Let $r \in \mathbb{R}$ be a fixed real number. Let $A \subseteq \mathbb{R}$ be a subset of real numbers. Let $B = \{x - r : x \in A\}$. Find a relation between $\sup(A)$ and $\sup(B)$, and prove it.

Claim: $\sup(A) - r = \sup(B)$.

Pf: We first prove $\sup(A) - r \geq \sup(B)$.

We know $\sup(A) \geq x \quad \forall x \in A$ [def. of $\sup(A)$].

$\Rightarrow \sup(A) - r \geq x - r \quad \forall x \in A$.

$\Rightarrow \sup(A) - r \geq y \quad \forall y \in B$.

$\Rightarrow \sup(A) - r \geq \sup(B)$ [def. of $\sup(B)$].

We next prove $\sup(A) - r \leq \sup(B)$.

We know $x \leq \sup(B) \quad \forall x \in B$ [def. of $\sup(B)$]

$\Rightarrow x + r \leq \sup(B) + r \quad \forall x \in B$

$\Rightarrow y \leq \sup(B) + r \quad \forall y \in A$

$\Rightarrow \sup(A) \leq \sup(B) + r$ [def. of $\sup(A)$].

$\therefore \sup(A) - r = \sup(B)$. ■

5. **(15 pts)** Let E be a subset of real numbers of the form

$$E = \{\frac{\sqrt{2}m}{n} : m \in \mathbb{Z}, n \in N\}.$$

Show that E is dense in $\mathbb{R}$. [Hint: Consider any interval (a, b). Manipulate this interval and use density of $\mathbb{Q}$]

Consider any (a,b) with. $a<b$

Consider the interval $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$,

$\exists\, y \in \mathbb{Q}$ s.t. $y \in \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

Also since $y \in \mathbb{Q}$, $\exists\, m \in \mathbb{Z}$ and $n \in \mathbb{N}$ s.t.

$$y = \frac{m}{n}.$$

$$\Rightarrow \quad \frac{m}{n} \in \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) \Rightarrow \frac{a}{\sqrt{2}} < \frac{m}{n} < \frac{b}{\sqrt{2}}$$

$$\Rightarrow a < \frac{\sqrt{2}m}{n} < b.$$

$$\Rightarrow \frac{\sqrt{2}m}{n} \in (a,b).$$

Since $\frac{\sqrt{2}m}{n} \in E$. We let $x = \frac{\sqrt{2}m}{n}$.

and $x \in E$ and $x \in (a,b)$. $\blacksquare$

6. **(10 pts)** Find

$$\lim_{n\to\infty} \frac{2n^2+1}{3n^2+2}$$

and justify your answer.

Claim: $\lim_{n\to\infty} \frac{2n^2+1}{3n^2+2} = \frac{2}{3}$

Pf: 1. Consider any $\epsilon > 0$.

2. Let N be any natural number larger than $\frac{1}{3\sqrt{\epsilon}}$ (by Archimedean principle)

3. Consider $n \geq N$.

4. $|s_n - L| = \left|\frac{2n^2+1}{3n^2+2} - \frac{2}{3}\right| = \left|\frac{-1}{3(3n^2+2)}\right|$

$= \frac{1}{3(3n^2+2)}$

Since $n \geq N \Rightarrow n^2 \geq N^2 > \frac{1}{9\epsilon}$

$\Rightarrow 3n^2 \geq \frac{1}{3\epsilon} \Rightarrow 3n^2+2 > \frac{1}{3\epsilon}$

$\Rightarrow \epsilon > \frac{1}{3(3n^2+2)}$

$\therefore |s_n - L| = \frac{1}{3(3n^2+2)} < \epsilon$ ■.

Scratch work:
(You don't have to show this on the test)

$\left|\frac{2n^2+1}{3n^2+2} - \frac{2}{3}\right| < \epsilon$

$\Rightarrow \left|\frac{-1}{3(3n^2+2)}\right| < \epsilon$

$\Rightarrow \frac{1}{3(3n^2+2)} < \epsilon$

$\Rightarrow \frac{1}{3\epsilon} < 3n^2+2$

$\Rightarrow \frac{1}{9\epsilon} - \frac{2}{3} < n^2$

choose

$N > \frac{1}{3\sqrt{\epsilon}}$.

7. **(10 pts)** Show that $\min\{x, y\} = \frac{(x+y)}{2} - \frac{|x-y|}{2}$.

We proceed by considering 2 cases :

Case 1 : $x \geq y$.

$\Rightarrow x - y \geq 0 \Rightarrow |x-y| = x-y$.

$$\Rightarrow \frac{x+y}{2} - \frac{|x-y|}{2} = \frac{x+y}{2} - \frac{(x-y)}{2}$$

$$= y.$$

Also $\min\{x,y\} = y$ since $x \geq y$.

Thus $\min\{x,y\} = y = \frac{x+y}{2} - \frac{|x-y|}{2}$

Case 2 : $x < y$.

$\Rightarrow x - y < 0 \Rightarrow |x-y| = y-x$.

$$\Rightarrow \frac{x+y}{2} - \frac{|x-y|}{2} = \frac{x+y}{2} - \frac{(y-x)}{2}$$

$$= x.$$

Also $\min\{x,y\} = x$ since $x < y$.

Thus $\min\{x,y\} = x = \frac{x+y}{2} - \frac{|x-y|}{2}$

In both cases $\min\{x,y\} = \frac{x+y}{2} - \frac{|x-y|}{2}$.