Let X be a set equipped with $(+, \cdot, <)$.

Field Axioms :

1. For any $a, b \in X$ there is an element $a + b \in X$ and $a + b = b + a$.
2. For any $a, b, c \in X$ the identity $(a + b) + c = a + (b + c)$ is true.
3. There is a unique number $0 \in X$ so that, for all $a \in X$, $a + 0 = 0 + a = a$.
4. For any number $a \in X$ there is a corresponding number denoted by a with the property that $a+(a) = 0$.
5. For any $a, b \in X$ there is an element $a \cdot b \in X$ and $a \cdot b = b \cdot a$.
6. For any $a, b, c \in X$ the identity $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ is true.
7. There is a unique element $1 \in X$ so that $a \cdot 1 = 1 \cdot a = a$ for all $a \in X$.
8. For any number $a \in X$, $a \neq 0$, there is a corresponding number denoted a^1 with the property that $a \cdot a^{-1} = 1$.
9. For any $a, b, c \in X$ the identity $(a + b) \cdot c = a \cdot c + b \cdot c$ is true.

Order Axioms :

1. For any $a, b \in X$ exactly one of the statements $a = b$, $a < b$ or $b < a$ is true.
2. For any $a, b, c \in X$ if $a < b$ is true and $b < c$ is true, then $a < c$ is true.
3. For any $a, b \in X$ if $a < b$ is true, then $a + c < b + c$ is also true for any $c \in X$.
4. For any $a, b \in X$ if $a < b$ is true, then $a \cdot c < b \cdot c$ is also true for any $c \in X$ for which $c > 0$.

Consequences from class for an ordered field X:

1. $0 \cdot a = 0$ for any $a \in X$.
2. $(a + b)^2 = a^2 + a \cdot b + a \cdot b + b^2$
3. $(-a) \cdot (b) = -(a \cdot b)$
4. $(a + b)(a + (-b)) = a^2 + (-b^2)$
5. $-(-a) = a$
6. $0 < x^2$ for any $x \in X$ such that $x \neq 0$
7. $0 < 1$
8. $0 < a$ if and only if $-a < 0$
9. If $a > 0$ and $b > 0$, then $a \cdot b > 0$. If $a < 0$ and $b > 0$ then $a \cdot b < 0$. If $a < 0$ and $b < 0$ then $a \cdot b > 0$.

1. **(5 pts)** (a) Let X be an ordered field and $E \subseteq X$. Define precisely the least upper bound for E.

$m \in X$ is a least upper bound for E

if $\forall x \in E$, $x \leq m$, and

$\forall m' \in X$ s.t. m' is an upper bound for E.

$m \leq m'$

(b)**(5 pts)** Write down the axiom of completeness for defining the real numbers.

Let X be an ordered field. Every nonempty subset $E \subseteq X$ that has an upper bound also has a least upper bound.

(c) **(5 pts)** Write the negation of the following statement :

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 5.$$

Negation: $\exists x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $x+y \neq 5$.

2. **(10 pts)** Consider the set $X = \mathbb{R} \times \mathbb{R}$, i.e., the set of all tuples (a, b) where $a, b \in \mathbb{R}$. We define two operations $+$ and $\cdot$ on this set as follows : $(a, b) + (a', b') = (a + a', b + b')$ and $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$. Clearly, $(0, 0)$ satisfies the field axiom 3. Also, $(1, 1)$ satisfies the field axiom 7. Is X equipped with these operations $(+, \cdot)$ a field ? Why or why not ?

Answer: $X = \mathbb{R} \times \mathbb{R}$ with $(+, \cdot)$ is NOT a field.

Justification: Consider the element $(1,0)$.

$\forall (a,b) \in X$,

$$(a,b)\cdot(1,0) = (a,0)$$

Thus there is no element $(a,b) \in X$ such that $(a,b)\cdot(1,0) = (1,1)$.

Note $(0,0)$ satisfies field axiom 3 and thus $(0,0)$ is the 0 element. $(1,1)$ satisfies the field axiom 7, it is the 1 element.

But then $(1,0)$ contradicts field axiom 8, since $(1,0) \neq (0,0)$ and yet there is no $(a,b) \in X$ such that $(a,b)\cdot(1,0) = (1,1)$.

3. **(10 pts)** Show that for all natural numbers $n \in \mathbb{N}$, the following identity holds for any real number $r \in \mathbb{R}$ such that $r \neq 1$.

$$1 + r + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

<u>Proof by induction</u> :

1. Establish $P(1)$ is true :

Need to show : $1 + r = \dfrac{r^{1+1} - 1}{r - 1}$

Right hand side $= \dfrac{r^2 - 1}{r - 1} = \dfrac{(r+1)(r-1)}{r-1}$

$= r + 1$

$=$ Left hand side.

2. Assume $P(n-1)$ is true, and show $P(n)$ is true.

$$1 + r + r^2 + \ldots + r^n = \underbrace{1 + r + r^2 + \ldots + r^{n-1}} + r^n$$

$$= \frac{r^{((n-1)+1)} - 1}{r - 1} + r^n$$

(by induction hypothesis).

$$= \frac{r^n - 1}{r - 1} + r^n$$

$$= \frac{r^n - 1 + r^n(r - 1)}{r - 1}$$

$$= r^{n+1} - 1$$

4. **(10 pts)** Let A, B be sets.

Prove that $A \subseteq B$ if and only if $B = (B \setminus A) \cup A$.

First show $A \subseteq B \Rightarrow B = (B \setminus A) \cup A$.

First show $B \subseteq (B \setminus A) \cup A$

Consider $x \in B$. Consider 2 exhaustive cases :

Case 1 : $x \in A \Rightarrow x \in A \cup (B \setminus A)$

Case 2 : $x \notin A \Rightarrow x \in B$ and $x \notin A$
$\Rightarrow x \in B \setminus A$
$\Rightarrow x \in (B \setminus A) \cup A$

In both cases $x \in (B \setminus A) \cup A$.

Next show $(B \setminus A) \cup A \subseteq B$.

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5. **(15 pts)** Show that $|\mathbb{Z}| = |\mathbb{N}|$. (You may, if you need, use the following facts without proof : (i) It $A \subseteq B$ then $|A| \leq |B|$, (ii) $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, (iii) Countable union of countable sets is countable)

Let $A_1 = \{x \in \mathbb{Z} : x > 0\}$.
$A_2 = \{0\}$.
$A_3 = \{x \in \mathbb{Z} : x < 0\}$.

$\therefore \mathbb{Z} = A_1 \cup A_2 \cup A_3$

First show A_1 is countable : $f : A_1 \rightarrow \mathbb{N}$
$f(x) = x$.

Since $x \in A_1$, $x \in \mathbb{Z}$ and $x > 0$
$\Rightarrow x \in \mathbb{N}$
$\Rightarrow f(x) = x \in \mathbb{N}$.

f is injective :

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6. Show the following for an ordered field X using the field axioms, order axioms and the consequences from class.

(a) **(7 pts)** Show that $a < b$ if and only if $-b < -a$.

$$a < b$$

$$\Rightarrow a + ((-a) + (-b)) < b + ((-a) + (-b)) \quad \text{[Order Axiom 3]}$$

$$\Rightarrow (a + (-a)) + (-b) < b + ((-b) + (-a)) \quad \text{[Field axioms 2 and 1]}$$

$$\Rightarrow 0 + (-b) < (b + (-b)) + (-a) \quad \text{[Field axioms 4 and 2]}$$

$$\Rightarrow -b < 0 + (-a) \quad \text{[Field axioms 3 and 4]}$$

$$\Rightarrow -b < -a \quad \text{[Field axiom 3].}$$

(b) **(7 pts)** Show that if $a < b$ and $c < d$ then $a + c < b + d$.

$$a < b$$

$$\Rightarrow a + c < b + c \quad \text{[Order axiom 3]}$$

Also $c < d$

$$\Rightarrow c + b < d + b \quad \text{[Order axiom 3].}$$

$$\Rightarrow b + c < b + d \quad \text{[Field axiom 1].}$$

$\therefore$ $a + c < b + c$ and $b + c < b + d$

$$\Rightarrow a + c < b + d \quad \text{[Order axiom 2]}$$

(c) **(6 pts)** Show that $a > 0$ if and only if $a^{-1} > 0$. (Recall that a^{-1} is an element in X such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$)

Show $a > 0 \Rightarrow a^{-1} > 0$.

By contradiction.

Assume $a^{-1} < 0$. $\Rightarrow a^{-1} a < 0 \cdot a$ [since $a > 0$ so we use order axiom 4]

$\Rightarrow 1 < 0$ [Field axiom 4 and consequence 1]

this contradicts Consequence 7.

Show $a^{-1} > 0 \Rightarrow a > 0$ ...

4 CONTD.

Show $(B \setminus A) \cup A \subseteq B$

Consider $x \in (B \setminus A) \cup A$

Consider 2 exhaustive cases:

Case 1: $x \in A$ and since $A \subseteq B$,

$\Rightarrow x \in B$

Case 2: $x \notin A$ and since $x \in (B \setminus A) \cup A$.

$\Rightarrow x \in B \setminus A$

$\Rightarrow x \in B$

In both cases $x \in B$

Next show $B = (B \setminus A) \cup A \Rightarrow A \subseteq B$.

Consider $x \in A$.

$\Rightarrow x \in A \cup (B \setminus A)$.

$\Rightarrow x \in B$ (since $B = A \cup (B \setminus A)$).

$\blacksquare$.

5 CONTD.

$$f(x) = f(y)$$

$$\Rightarrow x = y. \quad \left[\text{By definition } \begin{array}{l} f(x) = x \\ \text{and } f(y) = y \end{array}\right].$$

<u>A_2 is countable</u>: $f : A_2 \to \mathbb{N}$

$f(0) = 1$. is injective

<u>A_3 is countable</u>: $f : A_3 \to \mathbb{N}$

$$f(x) = -x.$$

Since $x \in A_3$, $x \in \mathbb{Z}$ and $x < 0$.

$$\Rightarrow -x \in \mathbb{N}.$$

$$\Rightarrow f(x) = -x \in \mathbb{N}.$$

Show f is injective:

$$f(x) = f(y)$$

$$\Rightarrow -x = -y \quad \left[\text{By def., } \begin{array}{l} f(x) = -x \\ f(y) = -y \end{array}\right]$$

$$\Rightarrow x = y$$

Thus, $A_1 \cup A_2 \cup A_3$ is countable since a countable (finite in this case) union of countable sets is countable.

Alternate proof of 5:

Define $f : \mathbb{Z} \to \mathbb{N}$ as follows:

$$f(x) = \begin{cases} 2x & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$$

First show $f(x) \in \mathbb{N}$ for all $x \in \mathbb{Z}$. :

If $x = 0$, then $f(x) = 1 \in \mathbb{N}$.

If $x > 0$, then $f(x) = 2x \in \mathbb{N}$.

If $x < 0$, then $f(x) = -2x + 1$

and since $-2x \in \mathbb{N}$ because $x \in \mathbb{Z}$ and $x < 0$, we have $-2x+1 \in \mathbb{N}$.

Next show $f(x)$ is injective:

Assume $f(x) = f(y)$

<u>Case 1</u>: $x = 0 \Rightarrow f(x) = 1$

$\Rightarrow f(y) = 1 \Rightarrow y = 0$ by definition of f

<u>Case 2</u>: $x > 0 \Rightarrow f(x) = 2x$

$\Rightarrow f(y) = 2x$

But then $y > 0$ since $y = 0 \Rightarrow f(y) = 1$.

and $y < 0 \Rightarrow f(y) = -2x+1$ which is odd.

$\Rightarrow 2x = f(y) = 2y \Rightarrow x = y.$

Case 3: $x < 0$

$\Rightarrow f(x) = -2x+1$

$\Rightarrow f(y) = -2x+1$

But then $y < 0$ since $y=0 \Rightarrow f(y) = 1$

and $-2x+1 > 0$ since $x < 0$.

and $y > 0 \Rightarrow f(y) = 2y$ which is even

but $-2x+1$ is odd.

$\Rightarrow -2x+1 = f(y) = -2y+1$

$\Rightarrow -2x = -2y$

$\Rightarrow x = y$.

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(d) **(5 pts)** Define $\overline{2} = 1 + 1$. Show that $(a+b)^2 = a^2 + \overline{2} \cdot a \cdot b + b^2$.

$$
\begin{aligned}
(a+b)^2 &= a^2 + ab + ab + b^2 && [\text{Consequence } 2(\text{i})]. \\
&= a^2 + 1 \cdot ab + 1 \cdot ab + b^2 && [\text{~~Field~~ axiom } 3]. \\
&= a^2 + (1+1)\, ab + b^2 && [\text{Field axiom } 9]. \\
&= a^2 + \overline{2} \cdot ab + b^2 && [\text{By def. } \overline{2} = 1+1].
\end{aligned}
$$

7. (a) **(5 pts)** Given an example of an ordered field X which does not satisfy the axiom of completeness. This means you have to give an example of a subset $E \subseteq X$ which has an upper bound, but does not have a least upper bound. You do not have to prove that a least upper bound does not exist.

$X = \mathbb{Q}$

$E = \{x \in \mathbb{Q} : x^2 < 2\}$.

E has many upper bounds (e.g. 4) but no least upper bound.

(b) **(10 pts)** Consider the following instances of an ordered field X and a subset E. State in each case if E has a maximum, minimum, least upper bound and greatest lower bound. If they exist, write the value. You do not have to justify your answers.

(i) $X = \mathbb{R}$. $E = \{x \in \mathbb{R} : 0 \leq x < 5\}$

No max, min = 0, least upper bound = 5, greatest lower bound = 0.

(ii) $X = \mathbb{Q}$. $E = \{x \in \mathbb{R} : 0 \leq x < 5\}$

No max, min = 0, least U.B. = 5, greatest L.B. = 0.