Let $X$ be a set equipped with $(+,\cdot,<)$.

**Field Axioms:**

1. For any $a, b \in X$ there is an element $a + b \in X$ and $a + b = b + a$.
2. For any $a, b, c \in X$ the identity $(a + b) + c = a + (b + c)$ is true.
3. There is a unique number $0 \in X$ so that, for all $a \in X$, $a + 0 = 0 + a = a$.
4. For any number $a \in X$ there is a corresponding number denoted by $a$ with the property that $a + (a) = 0$.
5. For any $a, b \in X$ there is an element $a \cdot b \in X$ and $a \cdot b = b \cdot a$.
6. For any $a, b, c \in X$ the identity $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ is true.
7. There is a unique element $1 \in X$ so that $a \cdot 1 = 1 \cdot a = a$ for all $a \in X$.
8. For any number $a \in X$, $a \neq 0$, there is a corresponding number denoted $a^{-1}$ with the property that $a \cdot a^{-1} = 1$.
9. For any $a, b, c \in X$ the identity $(a + b) \cdot c = a \cdot c + b \cdot c$ is true.

**Order Axioms:**

1. For any $a, b \in X$ exactly one of the statements $a = b$, $a < b$ or $b < a$ is true.
2. For any $a, b, c \in X$ if $a < b$ is true and $b < c$ is true, then $a < c$ is true.
3. For any $a, b \in X$ if $a < b$ is true, then $a + c < b + c$ is also true for any $c \in X$.
4. For any $a, b \in X$ if $a < b$ is true, then $a \cdot c < b \cdot c$ is also true for any $c \in X$ for which $c > 0$.

**Consequences from class for an ordered field $X$:**

1. $0 \cdot a = 0$ for any $a \in X$.
2. $(a + b)^2 = a^2 + a \cdot b + a \cdot b + b^2$
3. $(-a) \cdot (b) = -(a \cdot b)$
4. $(a + b)(a + (-b)) = a^2 + (-b^2)$
5. $-(-a) = a$
6. $0 < x^2$ for any $x \in X$ such that $x \neq 0$
7. $0 < 1$
8. $0 < a$ if and only if $-a < 0$
9. If $a > 0$ and $b > 0$, then $a \cdot b > 0$. If $a < 0$ and $b > 0$, then $a \cdot b < 0$. If $a < 0$ and $b < 0$ then $a \cdot b > 0$. 
1. (5 pts) (a) Let $X$ be an ordered field and $E \subseteq X$. Define precisely the least upper bound for $E$.

$m \in X$ is a least upper bound for $E$

if

\[ \forall x \in E, \quad x \leq m, \quad \text{and} \]

\[ \forall m' \in X \text{ s.t. } m' \text{ is an upper bound for } E. \]

\[ m \leq m' \]

(b) (5 pts) Write down the axiom of completeness for defining the real numbers.

Let $X$ be an ordered field. Every nonempty subset $E \subseteq X$ that has an upper bound also has a least upper bound.

(c) (5 pts) Write the negation of the following statement:

\[ \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 5. \]

Negation: \[ \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \neq 5. \]
2. (10 pts) Consider the set $X = \mathbb{R} \times \mathbb{R}$, i.e., the set of all tuples $(a, b)$ where $a, b \in \mathbb{R}$. We define two operations $+$ and $\cdot$ on this set as follows:

$(a, b) + (a', b') = (a + a', b + b')$ and $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$. Clearly, $(0, 0)$ satisfies the field axiom 3. Also, $(1, 1)$ satisfies the field axiom 7. Is $X$ equipped with these operations $(+, \cdot)$ a field? Why or why not?

Answer: $X = \mathbb{R} \times \mathbb{R}$ with $(+, \cdot)$ is NOT a field.

Justification: Consider the element $(1, 0)$.

Let $(a, b) \in X,

(a, b) \cdot (1, 0) = (a, 0)

Thus there is no element $(a, b) \in X$ such that $(a, b) \cdot (1, 0) = (1, 1)$.

Note $(0, 0)$ satisfies field axiom 3 and these $(0, 0)$ is the $0$ element.

$(1, 1)$ satisfies the field axiom 7, it is the $1$ element.

But then $(1, 0)$ contradicts field axiom 8, since $(1, 0) \neq (0, 0)$ and yet there is no $(a, b) \in X$ such that $(a, b) \cdot (1, 0) = (1, 1)$.
3. (10 pts) Show that for all natural numbers \( n \in \mathbb{N} \), the following identity holds for any real number \( r \in \mathbb{R} \) such that \( r \neq 1 \).

\[
1 + r + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}
\]

**Proof by induction:**

1. **Establish \( P(1) \) is true:**

   Need to show:

   \[
   1 + r = \frac{r^2 - 1}{r - 1}
   \]

   **Right hand side:**

   \[
   \frac{r^2 - 1}{r - 1} = \frac{(r+1)(r-1)}{r - 1} = r + 1
   \]

   = Left hand side.

2. **Assume \( P(n-1) \) is true, and show \( P(n) \) is true.**

   \[
   1 + r + r^2 + \ldots + r^n = 1 + r + r^2 + \ldots + r^{n-1} + r^n
   \]

   \[
   = \frac{(r^{n-1})(r^1 + 1)}{r - 1} + r^n
   \]

   (by induction hypothesis)

   \[
   = \frac{r^n - 1}{r - 1} + r^n
   \]

   \[
   = \frac{r^n - 1 + r^n(r - 1)}{r - 1}
   \]

   \[
   = \frac{r^{n+1} - 1}{r - 1}
   \]
4. (10 pts) Let \( A, B \) be sets.

Prove that \( A \subseteq B \) if and only if \( B = (B \setminus A) \cup A \).

First show \( A \subseteq B \implies B = (B \setminus A) \cup A \).

First show \( B \subseteq (B \setminus A) \cup A \).

Consider \( x \in B \). Consider 2 exhaustive cases:

Case 1: \( x \in A \implies x \in A \cup (B \setminus A) \).

Case 2: \( x \notin A \implies x \in B \setminus A \implies x \in (B \setminus A) \cup A \).

In both cases \( x \in (B \setminus A) \cup A \).

Next show \( (B \setminus A) \cup A \subseteq B \).

5. (15 pts) Show that \( |\mathbb{Z}| = |\mathbb{N}| \). (You may, if you need, use the following facts without proof: (i) If \( A \subseteq B \) then \( |A| \leq |B| \), (ii) \( |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| \), (iii) Countable union of countable sets is countable.)

Let \( A_1 = \{ x \in \mathbb{Z} : x > 0 \} \).

\( A_2 = \{ 0 \} \).

\( A_3 = \{ x \in \mathbb{Z} : x < 0 \} \).

\( \mathbb{Z} = A_1 \cup A_2 \cup A_3 \).

First show \( A_1 \) is countable:

\( f : \emptyset \to \mathbb{N} \)

where \( f(x) = x \).

Since \( x \in A_1 \), \( x \in \mathbb{Z} \) and \( x > 0 \)

\( \implies x \in \mathbb{N} \)

\( \implies f(x) = x \in \mathbb{N} \).

\( f \) is injective.

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6. Show the following for an ordered field $X$ using the field axioms, order axioms and the consequences from class.

(a) (7 pts) Show that $a < b$ if and only if $-b < -a$.

\[
\begin{align*}
    a < b & \\
    \Rightarrow a + (-a) + (-b) & < b + (-a) + (-b) \quad \text{[Order axiom 3]} \\
    \Rightarrow (a + (-a)) + (-b) & < b + (-b) + (-a) \quad \text{[Field axioms 2 and 1]} \\
    \Rightarrow 0 + (-b) & < (b + (-b)) + (-a) \quad \text{[Field axioms 4 and 2]} \\
    \Rightarrow -b & < 0 + (-a) \quad \text{[Order axioms 3 and 4]} \\
    \Rightarrow -b & < -a \quad \text{[Field axiom 3]}
\end{align*}
\]

(b) (7 pts) Show that if $a < b$ and $c < d$ then $a + c < b + d$.

\[
\begin{align*}
a < b & \\
\Rightarrow a + c & < b + c \quad \text{[Order axiom 3]} \\
\text{Also: } c < d & \\
\Rightarrow c + b & < d + b \quad \text{[Order axiom 3]} \\
\Rightarrow b + c & < b + d \quad \text{[Field axiom 1]}
\end{align*}
\]
\[
\therefore a + c < b + c \quad \text{and} \quad b + c < b + d \\
\Rightarrow a + c < b + d \quad \text{[Order axiom 2]}
\]

(c) (6 pts) Show that $a > 0$ if and only if $a^{-1} > 0$. (Recall that $a^{-1}$ is an element in $X$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$)

By contradiction, assume $a^{-1} < 0$. Then $a^{-1}a < 0 \cdot a$ since $a > 0$.

\[
\begin{align*}
    \Rightarrow 1 & < 0 \quad \text{[Field axiom 1 and consequence 4]}
\end{align*}
\]

This contradicts consequence 7. Show $a^{-1} > 0$.
Contd. 

Show \((B \setminus A) \cup A \subseteq B\)

Consider \(x \in (B \setminus A) \cup A\)

Consider 2 exhaustive cases:

Case 1: \(x \in A\) and since \(A \subseteq B\),
\[\Rightarrow x \in B\]

Case 2: \(x \notin A\) and since \(x \in (B \setminus A) \cup A\),
\[\Rightarrow x \in B \setminus A\]
\[\Rightarrow x \in B\]

In both cases \(x \in B\)

Next show \(B = (B \setminus A) \cup A \Rightarrow A \subseteq B\).

Consider \(x \in A\).
\[\Rightarrow x \in A \cup (B \setminus A)\]
\[\Rightarrow x \in B\] (since \(B = A \cup (B \setminus A)\)).

\(\Box\).
$f(x) = f(y)$
$\Rightarrow x = y$. \ [By definition, \( f(x) = x \) and \( f(y) = y \).]

$A_2$ is countable: $f: \emptyset A_2 \rightarrow \mathbb{N}$
$f(0) = 1$. is injective

$A_3$ is countable: $f: A_3 \rightarrow \mathbb{N}$
$f(x) = -x$.
Since $x \in A_3$, $x \in \mathbb{Z}$ and $x < 0$.
$\Rightarrow -x \in \mathbb{N}$.
$\Rightarrow f(x) = -x \in \mathbb{N}.
$
Show $f$ is injective.

$f(x) = f(y)$
$\Rightarrow -x = -y$ \ [by def., \( f(x) = -x \) \( f(y) = -y \)]
$\Rightarrow x = y$

Thus, $A_1 \cup A_2 \cup A_3$ is countable since a countable (finite in this case) union of countable sets is countable.
Alternate proof of 5:

Define \( f : \mathbb{Z} \to \mathbb{N} \) as follows:

\[
 f(x) = \begin{cases} 
 2x & \text{if } x > 0 \\
 1 & \text{if } x = 0 \\
 -2x + 1 & \text{if } x < 0
\end{cases}
\]

First show \( f(x) \in \mathbb{N} \) for all \( x \in \mathbb{Z} \):

- If \( x = 0 \), then \( f(x) = 1 \in \mathbb{N} \).
- If \( x > 0 \), then \( f(x) = 2x \in \mathbb{N} \).
- If \( x < 0 \), then \( f(x) = -2x + 1 \)

and since \(-2x \in \mathbb{N} \) because \( x \in \mathbb{Z} \) and \( x < 0 \),
we have \(-2x + 1 \in \mathbb{N} \).

Next show \( f(x) \) is injective:

Assume \( f(x) = f(y) \)

Case 1: \( x = 0 \) \( \Rightarrow \) \( f(x) = 1 \)
\( \Rightarrow \) \( f(y) = 1 \) \( \Rightarrow \) \( y = 0 \) by definition of \( f \).

Case 2: \( x > 0 \) \( \Rightarrow \) \( f(x) = 2x \)
\( \Rightarrow \) \( f(y) = 2x \)

But then \( y > 0 \) since \( y = 0 \) \( \Rightarrow \) \( f(y) = 1 \).

and \( y < 0 \) \( \Rightarrow \) \( f(y) = -2x + 1 \)
which is odd.

\( \Rightarrow \) \( 2x = f(y) = 2y \) \( \Rightarrow \) \( x = y \).
Case 3: \( x < 0 \)

\[ \Rightarrow f(x) = -2x + 1 \]

\[ \Rightarrow f(y) = -2x + 1 \]

But then \( y < 0 \) since \( y = 0 \Rightarrow f(y) = 1 \).

\[ \text{and } -2x + 1 > 0 \text{ since } x < 0. \]

\[ \text{and } y > 0 \Rightarrow f(y) = 2y \text{ which is even} \]

\[ \text{but } -2x + 1 \text{ is odd}. \]

\[ \Rightarrow -2x + 1 = f(y) = -2y + 1 \]

\[ \Rightarrow -2x = -2y \]

\[ \Rightarrow x = y. \]

\( \checkmark \)
(d) (5 pts) Define \( \overline{2} = 1 + 1 \). Show that \((a + b)^2 = a^2 + 2 \cdot a \cdot b + b^2\).

\[
(a+b)^2 = a^2 + ab + ab + b^2 \quad \text{[Consequence 2(i)]}
= a^2 + 1 \cdot ab + 1 \cdot ab + b^2 \quad \text{[Field axiom 3]}
= a^2 + (1+1) ab + b^2 \quad \text{[Field axiom 9]}
= a^2 + \overline{2} \cdot ab + b^2 \quad \text{[by def. \( \overline{2} = 1+1 \)]}
\]

7. (a) (5 pts) Given an example of an ordered field \( X \) which does not satisfy the axiom of completeness. This means you have to give an example of a subset \( E \subseteq X \) which has an upper bound, but does not have a least upper bound. You do not have to prove that a least upper bound does not exist.

\[
X = \mathbb{Q}
\]

\[
E = \left\{ \frac{x}{2} \in \mathbb{Q} : x^2 < 2 \right\}
\]

\( E \) has many upper bounds (e.g. 4) but no least upper bound.

(b) (10 pts) Consider the following instances of an ordered field \( X \) and a subset \( E \). State in each case if \( E \) has a maximum, minimum, least upper bound and greatest lower bound. If they exist, write the value. You do not have to justify your answers.

(i) \( X = \mathbb{R} \). \( E = \{x \in \mathbb{R} : 0 \leq x < 5\} \)

\( \text{No max}, \text{ min} = 0, \text{ least upper bound} = 5, \text{ greatest lower bound} = 0 \).

(ii) \( X = \mathbb{Q} \). \( E = \{x \in \mathbb{R} : 0 \leq x < 5\} \)

\( \text{No max}, \text{ min} = 0, \text{ least U.B.} = 5, \text{ greatest L.B} = 0 \).