Properties of absolute value function :

1. $|x| \geq 0 \quad \forall x \in \mathbb{R}$

2. $|x| < r$ if and only if $-r < x < r \quad \forall x, r \in \mathbb{R}$

3. $|-x| = |x| \quad \forall x \in \mathbb{R}$

4. $-|x| \leq x \leq |x| \quad \forall x \in \mathbb{R}$

5. $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$

6. $|x+y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$

7. $|x| - |y| \leq |x-y| \quad \forall x, y \in \mathbb{R}$ and $|y| - |x| \leq |x-y| \quad \forall x, y \in \mathbb{R}$

You can use the following fact without proof :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

1. (a) **(5 pts)** Give the precise definition of the limit of a sequence $\{s_n\}_{n=1}^{\infty}$.

$L \in \mathbb{R}$ is the limit of the sequence $\{s_n\}_{n=1}^{\infty}$

if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \quad |s_n - L| < \epsilon$.

Also acceptable: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}$

$(n \geq N) \Rightarrow |s_n - L| < \epsilon$.

(b) **(5 pts)** Show that $|x - a| < \epsilon$ if and only if $a - \epsilon < x < a + \epsilon$.

Using property 2 of 1.1:

$$|x - a| < \epsilon$$

$$\Leftrightarrow -\epsilon < x - a < \epsilon$$

$$\Leftrightarrow a - \epsilon < x < a + \epsilon$$

$\blacksquare$

2. **(10 pts)** Let $f(x) = (\frac{1}{2})^x$. Find the infimum of the set

$$E = \{f(x) : x \geq 0\}$$

and justify your answer.

Since $f(x) = \left(\frac{1}{2}\right)^x \geqslant 0$, 0 is a lower bound.

<u>claim:</u> $\inf E = 0$.

Pf: We use the theorem that if L is a lower bound, $L = \inf E \iff \forall \epsilon > 0, \exists y \in E, y < L + \epsilon$.

Consider arbitrary $\epsilon > 0$.

If $\epsilon \geqslant 2$. Choose $x = 0$, and let

$$y = f(x) = \left(\frac{1}{2}\right)^0 = 1.$$

Since $\epsilon \geqslant 2$.

$$L + \epsilon = 0 + \epsilon \geqslant 2 > 1 = y.$$

If $\epsilon < 2$. Choose $x = \dfrac{-\ln(\epsilon/2)}{\ln 2}$.

set $y = f(x) = \left(\frac{1}{2}\right)^{-\frac{\ln(\epsilon/2)}{\ln 2}} = \left(\frac{1}{2}\right)^{-\ln_2(\epsilon/2)}$

$$= 2^{\ln_2(\epsilon/2)} = \epsilon/2.$$

$$\therefore \quad y = \epsilon/2 < 0 + \epsilon = L + \epsilon.$$

Also, since $\epsilon < 2$, $x \geq 0$. Thus $y \in E$.

3. **(15 pts)** Show that the set of irrational numbers is dense in $\mathbb{R}$, i.e. $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$.

Consider any arbitrary (a, b) interval.

We construct $y \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $y \in (a,b)$.

We consider the interval $(a-\sqrt{2}, b-\sqrt{2})$.

We use the theorem : $\mathbb{Q}$ is dense in $\mathbb{R}$.

$\Rightarrow \exists\, m \in \mathbb{Z},\ n \in \mathbb{N}$ s.t.

$$\frac{m}{n} \in (a-\sqrt{2}, b-\sqrt{2}).$$

$$\Rightarrow a-\sqrt{2} < \frac{m}{n} < b-\sqrt{2}$$

$$\Rightarrow a < \frac{m}{n}+\sqrt{2} < b.$$

We set $y = \frac{m}{n}+\sqrt{2}$. Thus. $y \in (a,b)$.

Also y is irrational because otherwise $y - \frac{m}{n} = \sqrt{2}$ is rational which is a contradiction.

Thus. $y \in \mathbb{R}\setminus\mathbb{Q}$ and $y \in (a,b)$.

$\blacksquare$

4. **(15 pts)** Let $r > 0$ be a fixed positive real number. Let $A \subseteq \mathbb{R}$ be a subset of real numbers. Let $B = \{x \cdot r : x \in A\}$. Find a relation between $\sup(A)$ and $\sup(B)$, and prove it.

<u>Claim</u>: $\sup(A) \cdot r = \sup(B)$

Pf: We first show $\sup(A) \cdot r \leq \sup(B)$

$$x \leq \sup(B) \quad \forall x \in B.$$

[by def. of $\sup(B)$].

$$\Rightarrow \frac{x}{r} \leq \frac{\sup(B)}{r} \quad \forall x \in B.$$

$$\Rightarrow y \leq \frac{\sup(B)}{r} \quad \forall y \in A.$$

$$\Rightarrow \sup(A) \leq \frac{\sup(B)}{r} \quad [\text{def. of } \sup(A)].$$

$$\Rightarrow r \cdot \sup(A) \leq \sup(B).$$

We next show $\sup(A) \cdot r \geq \sup(B)$.

$$x \leq \sup(A) \quad \forall x \in A \quad [\text{def. of } \sup(A)].$$

$$\Rightarrow x \cdot r \leq \sup(A) \cdot r \quad \forall x \in A$$

$$\Rightarrow y \leq \sup(A) \cdot r \quad \forall y \in B.$$

$$\Rightarrow \sup(B) \leq \sup(A) \cdot r \quad [\text{def. of } \sup(B)].$$

Thus, $\sup(A) \cdot r = \sup(B)$.

$\blacksquare$

5. **(10 pts)** Let A and B be nonempty sets of real numbers such that every element of A is less than or equal to every element of B. Thus, $\forall x \in A, \forall y \in B$, $x \leq y$. Show that $\sup(A) \leq \inf(B)$. [Hint: For any arbitrary $x \in A$, we know that $x \leq y \;\; \forall y \in B$. Does this give a relation between x and $\inf(B)$?]

For an arbitrary $x \in A$.

$$x \leq y \quad \forall y \in B.$$

$$\Rightarrow x \leq \inf(B) \quad [\text{def. of } \inf(B)].$$

Since this is true for an arbitrary $x \in A$,

$$x \leq \inf(B) \quad \forall x \in A.$$

(i.e., $\inf(B)$ is an upper bound for A).

$$\Rightarrow \sup(A) \leq \inf(B) \quad [\text{def. of } \sup(A)].$$

$\blacksquare$.

6. Decide if the following sequences converge or diverge. Justify your answers.

(a) **(10 pts)**

$$s_n = \frac{2n^2+1}{3n^2+2}.$$

The sequence converges, $\lim_{n\to\infty} s_n = \frac{2}{3}$.

1. Consider arbitrary $\epsilon > 0$.

2. Choose N as a natural number $\geq \frac{1}{3\sqrt{\epsilon}}$. [Archimedean principle].

3. Consider $n \geq N$.

$$\Rightarrow \frac{1}{n} \leq \frac{1}{N} \Rightarrow \frac{1}{n^2} \leq \frac{1}{N^2}$$

$$\Rightarrow \frac{1}{n^2} \leq \frac{1}{N^2} \leq 9\epsilon \quad \left[N \geq \frac{1}{3\sqrt{\epsilon}}\right]$$

$$\Rightarrow \frac{1}{3n^2} \leq 3\epsilon$$

$$\Rightarrow \frac{1}{3n^2+2} < 3\epsilon \Rightarrow \frac{1}{3(3n^2+2)} < \epsilon.$$

Scratch Work (no need to show on answer sheet).

$$\left|\frac{2n^2+1}{3n^2+2} - \frac{2}{3}\right| = \left|\frac{-1}{3(3n^2+2)}\right|$$

$$= \frac{1}{3(3n^2+2)} < \epsilon$$

$$\Rightarrow \frac{1}{3\epsilon} < 3n^2+2$$

$$\Rightarrow \frac{1}{9\epsilon} - \frac{2}{3} < n^2$$

Select $N \geq \frac{1}{3\sqrt{\epsilon}}$

But $\left|s_n - \frac{2}{3}\right| = \left|\frac{2n^2+1}{3n^2+2} - \frac{2}{3}\right| = \left|\frac{-1}{3(3n^2+2)}\right| = \frac{1}{3(3n^2+2)} < \epsilon$

$\blacksquare$.

(b)**(10 pts)**

$$s_n = (-1)^n \cdot 2.$$

s_n diverges. We show $\forall L \in \mathbb{R}$, $\lim_{n\to\infty} s_n \neq L$.

Pf: 1. Consider an arbitrary $L \in \mathbb{R}$.

2. Let $\epsilon = 1/2$.

3. Consider $N \in \mathbb{N}$.

4. If $L \geq 0$, choose $n = 2N+1$.
   If $L < 0$, choose $n = 2N$.

5. Since $n = 2N$ or $n = 2N+1$,
   $n \geq N$.

To show: $|s_n - L| \geq \epsilon = 1/2$.

Case 1: $L \geq 0$. $|s_n - L| = |-2-L|$
$= 2 + L \geq 2 \geq 1/2$

Case 2: $L < 0$. $|s_n - L| = |2-L| = 2 - L$
$\geq 2 \geq 1/2$
since $L < 0$.

$\blacksquare$