1. **(6 pts)** (a) Let X be an ordered field and $E \subseteq X$. Define precisely the maximum element of E. Does a maximum element always exist ? Justify your answer.

The maximum element of E is an element

$M \in E$ such that

$\forall x \in E \quad x \leq M$.

A maximum element does not always exist

e.g. $X = \mathbb{R}$, $E = (0,1)$

E has no maximum element.

(b) **(7 pts)** Write the negation of the following statement :

$$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x \cdot y = 0.$$

State if the original statement is true or false, and if the negation is true or false. Justity your answer.

Negation : $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x \cdot y \neq 0$.

Original statement is true : $x = 0$ works.

Negation is false : $x = 0$.

There does not exist $y \in \mathbb{R}$

s.t. $x \cdot y \neq 0$.

(c) **(2 pts)** Write the statement "Sets A and B are disjoint." using set notation. Your answer should not contain any words in English.

$A \cap B = \emptyset$

2. **(10 pts)** Consider the set $X = \mathbb{R} \times \mathbb{R}$, i.e., the set of all tuples (a, b) where $a, b \in \mathbb{R}$. We define two operations $+$ and $\cdot$ on this set as follows : $(a, b) + (a', b') = (a + a', b + b')$ and $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$. Clearly, $(0, 0)$ satisfies the field axiom 3. Also, $(1, 1)$ satisfies the field axiom 7. Is X equipped with these operations $(+, \cdot)$ a field ? Why or why not ?

X is NOT a field.

Justification: Since $(0,0)$ satisfies axiom 3, $(0,0)$ is the 0 element.

Since $(1,1)$ satisfies field axiom 7, $(1,1)$ is the 1 element.

Consider the element $(1,0)$. Clearly $(1,0) \neq (0,0)$.

However, $(1,0)\cdot(a,b) = (a,0)$ for all $(a,b) \in X$.

$\therefore$, there does not exist any element (a,b) such that $(1,0)\cdot(a,b)$ is equal to $(1,1)$ which is the 1 element.

This contradicts field axiom 8.

3. **(10 pts)** Show that for all natural numbers $n \in \mathbb{N}$, the following inequality holds for any real number $x \in \mathbb{R}$ such that $x > 0$. (You do not have to justify everything using the complete ordered field axioms, i.e., you can use the usual rules of manipulating the real numbers without justifying them using the axioms)

$$(1+x)^n \geq 1 + nx$$

Proof by induction on n

1. Establish $P(1)$ is true.

Left hand side : $(1+x)^1 = 1+x$ Right hand side : $1+(1)(x)$
$= 1+x$

Left hand side $\geq$ Right hand side.
Done.

2. Assume $P(n-1)$ is true, and show $P(n)$ is true.

~~Consider~~ Assume $(1+x)^{n-1} \geq 1+(n-1)x$

Now, $(1+x)^n = (1+x)^{n-1}(1+x)$

$\geq (1+(n-1)x)(1+x)$ by induction hypothesis.

$= 1+(n-1)x + x + (n-1)x^2$

$= 1 + nx + (n-1)x^2$

Since $(n-1)x^2 \geq 0$

$1+nx+(n-1)x^2 \geq 1+nx.$

$\therefore (1+x)^n \geq 1+nx$

4. **(15 pts)** Let A_1, A_2 be arbitrary sets. Show that there exist two sets B_1 and B_2 such that they satisfy *both* of the following conditions. Justify every step.

(i) $B_1 \cap B_2 = \emptyset$.

(ii) $B_1 \cup B_2 = A_1 \cup A_2$.

Let $B_1 = A_1$ and $B_2 = A_2 \setminus A_1$.

1. Show (i) holds.

Consider $x \in B_2$ and show $x \notin B_1$

$x \in B_2 = A_2 \setminus A_1$

$\Rightarrow x \in A_2$ and $x \notin A_1$

$\Rightarrow x \notin A_1 = B_1$ ✓.

2. Show (ii) holds.

First show $(B_1 \cup B_2) \subseteq (A_1 \cup A_2)$.

Consider $x \in B_1 \cup B_2$

Consider 2 exhaustive cases:

Case 1: $x \in B_1 \Rightarrow x \in A_1$ (since $B_1 = A_1$)

$\Rightarrow x \in A_1 \cup A_2$

Case 2: $x \in B_2 \Rightarrow x \in A_2 \setminus A_1$ (since $B_2 = A_2 \setminus A_1$)

$\Rightarrow x \in A_2$ and $x \notin A_1$

$\Rightarrow x \in A_2 \cup A_1$

In both cases, $x \in A_2 \cup A_1$ ✓.

Next show $A_1 \cup A_2 \subseteq B_1 \cup B_2$

Consider $x \in A_1 \cup A_2$

Consider 2 cases:

Case 1: $x \in A_1$

$\Rightarrow x \in B_1$ (since $B_1 = A_1$)

$\Rightarrow x \in B_1 \cup B_2$

Case 2: $x \notin A_1$

since $x \in A_1 \cup A_2$ and $x \notin A_1$

$\Rightarrow x \in A_2$ and $x \notin A_1$

$\Rightarrow x \in A_2 \setminus A_1$

$\Rightarrow x \in B_2$ (since $B_2 = A_2 \setminus A_1$)

$\Rightarrow x \in B_2 \cup B_1$

In both cases, $x \in B_1 \cup B_2$

5. **(15 pts)** Let A and B be two countable sets. Show that $A \times B$ is countable. (You may, if you need, use the following facts without proof : (i) If $A \subseteq B$ then $|A| \leq |B|$, (ii) $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, (iii) Countable union of countable sets is countable)

Since $|A| \leq |\mathbb{N}|$ and $|B| \leq |\mathbb{N}|$ (both are countable)

$\exists$ an injective map $f : A \to \mathbb{N}$
and an injective map $g : B \to \mathbb{N}$.

We now construct an injective map $h : A \times B \to \mathbb{N} \times \mathbb{N}$.

$$h((a,b)) = (f(a), g(b))$$

1. Show $\forall (a,b) \in A \times B$, $h((a,b)) \in \mathbb{N} \times \mathbb{N}$

since $h((a,b)) = (f(a), g(b))$
and $f(a), g(b) \in \mathbb{N}$ by definition of $f + g$,
$h((a,b)) = (f(a), g(b)) \in \mathbb{N} \times \mathbb{N}$.

2. Show h is injective: Show that $h((a_1,b_1)) = h((a_2,b_2)) \Rightarrow (a_1,b_1) = (a_2,b_2)$

$h((a_1,b_1)) = (f(a_1), g(b_1))$ and $h((a_2,b_2)) = (f(a_2), g(b_2))$

$h((a_1,b_1)) = h((a_2,b_2)) \Rightarrow (f(a_1), g(b_1)) = (f(a_2), g(b_2))$
$\Rightarrow f(a_1) = f(a_2)$ and $g(b_1) = g(b_2)$
$\Rightarrow a_1 = a_2$ and $b_1 = b_2$
(since $f + g$ are injective)
$\Rightarrow (a_1, b_1) = (a_2, b_2)$ $\blacksquare$.

$\therefore |A \times B| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| \Rightarrow |A \times B| \leq |\mathbb{N}|$.

6. Show the following are true for an ordered field X using the field axioms, order axioms and the consequences from class. Justify every step with a reference to an axiom or consequence. You can use the numbering on the cover sheet : for example, "blah-blah [Order axiom 3]", "blah-blah [Field Axiom 7], "blah-blah [Consequence 2(ii)].

(a) **(6 pts)** Show that $-(a+b) = (-a) + (-b)$.

Consider $(a+b) + ((-a)+(-b)) = (a+b) + ((-b)+(-a))$ [Field axiom 1]

$= ((a+b)+(-b)) + (-a)$ [Field axiom 2]

$= (a + (b+(-b))) + (-a)$ [Field axiom 2]

$= (a+0) + (-a)$ [Field axiom 4].

$= a + (-a)$ [Field axiom 3]

$= 0$.

Also $(a+b) + ((-a)+(-b)) = ((-a)+(-b)) + (a+b)$ [Field axiom 1]

$\Rightarrow (a+b) + ((-a)+(-b)) = ((-a)+(-b)) + (a+b) = 0$.

But, $(a+b) + (-(a+b)) = (-(a+b)) + (a+b) = 0$ [by definition of $-(a+b)$]

By consequence 4, $-(a+b) = (-a)+(-b)$

(b) **(7 pts)** Show that if $a < b$ and $c < d$, and $0 < b$ and $0 < c$, then $a \cdot c < b \cdot d$.

$a < b$

$\Rightarrow a \cdot c < b \cdot c$ [Since $0 < c$ and Order axiom 4]

Also $c < d$

$\Rightarrow c \cdot b < d \cdot b$ [Since $0 < b$ and order axiom 4.]

$\Rightarrow b \cdot c < b \cdot d$ [Field axiom 5].

$\therefore a \cdot c < b \cdot c$ and $b \cdot c < b \cdot d$

$\Rightarrow a \cdot c < b \cdot d$ [Order axiom 2].

(c) **(7 pts)** Show that if $0 < a$, $0 < b$ and $b^2 < a^2$ then $b < a$. [Hint: Use Consequences 2(ii) and 9]

$b^2 < a^2 \Rightarrow b^2 + (-b^2) < a^2 + (-b^2)$ [Order axiom 3]

$\Rightarrow 0 < (a+b)(a+(-b))$ [Field axiom 3 and Consequence 2(ii)]

Proof by contradiction: Suppose $a < b$.

$\Rightarrow a + (-b) < b + (-b)$ [Order axiom 3]

$\Rightarrow a + (-b) < 0$ [Field axiom 3]

Also $0 < a \Rightarrow 0 + b < a + b$ [Order axiom 3].

$\Rightarrow b < a+b$.

and $0 < b$

$\Rightarrow 0 < a+b$.

Thus by Consequence 9, $(a+(-b)) \cdot (a+b) < 0$

This is a contradiction to $0 < (a+b)(a+(-b))$

7. (a) **(5 pts)** Given an example of an ordered field X and a subset $E \subseteq X$ such that E has a *lower* bound, but does not have a greatest lower bound. You do not have to prove that a greatest lower bound does not exist, the example will suffice.

$X = \mathbb{Q}$

$E = \{ x \in \mathbb{Q} : 2 < x^2 \}$.

(b) **(10 pts)** Consider the following instances of an ordered field X and a subset E. State in each case if E has a maximum, minimum, least upper bound and greatest lower bound. If they exist, write the value. You do not have to justify your answers.

(i) $X = \mathbb{R}$. $E = \{x \in \mathbb{R} : -1 < x \leq 2\}$

Max = 2

No min.

L.U.B = 2

G.L.B = -1.

(ii) $X = \mathbb{Q}$. $E = \{x \in \mathbb{Q} : -1 < x \leq 2\}$

Max = 2

No min.

L.U.B = 2

G.L.B = -1.