Homework Problems : WEEK IX

For any sequence $\{x_n\}_{n=1}^{\infty}$, we say $\beta \in \mathbb{R}$ is an *eventual upper bound* of $\{x_n\}_{n=1}^{\infty}$ if there exists $N \in \mathbb{N}$ such that $x_n < \beta$ for all $n \geq N$. Similarly, we say $\alpha \in \mathbb{R}$ is an *eventual lower bound* of $\{x_n\}_{n=1}^{\infty}$ if there exists $N \in \mathbb{N}$ such that $\alpha < x_n$ for all $n \geq N$.

Let $\mathcal{U}$ be the set of all eventual upper bounds of $\{x_n\}_{n=1}^{\infty}$ and $\mathcal{L}$ be the set of all eventual lower bounds of $\{x_n\}_{n=1}^{\infty}$, i.e.,

$$\mathcal{U} = \{\beta \in \mathbb{R} : \exists N \in \mathbb{N} \text{ s.t. } x_n < \beta \quad \forall n \geq N\}$$

$$\mathcal{L} = \{\alpha \in \mathbb{R} : \exists N \in \mathbb{N} \text{ s.t. } \alpha < x_n \quad \forall n \geq N\}$$

We define $\limsup x_n = \inf \mathcal{U}$ and $\liminf x_n = \sup \mathcal{L}$.

1. Do 2.12.2, 2.12.3, 2.12.4, 2.12.5, 2.12.8.

2. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence.

   (i) Show that $\limsup x_n \neq \infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded above. Find an example where x_n is bounded above and $\limsup x_n = -\infty$. (Notice that the claim I made in class is not entirely correct)

   (ii) Show that $\liminf x_n \neq -\infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded below. Find an example where x_n is bounded below and $\liminf x_n = \infty$. (Notice that the claim I made in class is not entirely correct)

3. Prove that if $\{x_n\}_{n=1}^{\infty}$ converges, then $\liminf x_n = \lim_{n\to\infty} x_n = \limsup x_n$.

4. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence that is bounded. Define a new sequence as follows :

   $$t_n = \sup\{x_m : m \geq n\} = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}.$$

   (i) Show that t_n is a monotone sequence.

   (ii) Show that t_n is bounded.

   (iii) Show that t_n converges and $\lim_{n\to\infty} t_n = \limsup x_n$.

   The above exercise justifies the name "lim sup".

5. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence that is bounded. Define a new sequence as follows :

   $$s_n = \inf\{x_m : m \geq n\} = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}.$$

   (i) Show that s_n is a monotone sequence.

   (ii) Show that s_n is bounded.

   (iii) Show that s_n converges and $\lim_{n\to\infty} s_n = \liminf x_n$.

   The above exercise justifies the name "lim inf".

6. Do 2.13.4, 2.13.5, 2.13.9, 2.13.10, 2.13.11 (for the last 3 problems, Exercises 4 and 5 could be useful).

7. Do 3.4.3, 3.4.4, 3.4.5, 3.4.6 (for this one you might need the result from the previous HW for problem 2.11.6 which showed that every subsequence of a convergent sequence converges), 3.4.11, 3.4.14, 3.4.15, 3.4.17, 3.4.21