Homework Problems : WEEK IX

For any sequence \( \{x_n\}_{n=1}^{\infty} \), we say \( \beta \in \mathbb{R} \) is an eventual upper bound of \( \{x_n\}_{n=1}^{\infty} \) if there exists \( N \in \mathbb{N} \) such that \( x_n < \beta \) for all \( n \geq N \). Similarly, we say \( \alpha \in \mathbb{R} \) is an eventual lower bound of \( \{x_n\}_{n=1}^{\infty} \) if there exists \( N \in \mathbb{N} \) such that \( \alpha < x_n \) for all \( n \geq N \).

Let \( \mathcal{U} \) be the set of all eventual upper bounds of \( \{x_n\}_{n=1}^{\infty} \) and \( \mathcal{L} \) be the set of all eventual lower bounds of \( \{x_n\}_{n=1}^{\infty} \), i.e.,

\[
\mathcal{U} = \{ \beta \in \mathbb{R} : \exists N \in \mathbb{N} \text{ s.t. } x_n < \beta \, \forall n \geq N \}
\]
\[
\mathcal{L} = \{ \alpha \in \mathbb{R} : \exists N \in \mathbb{N} \text{ s.t. } \alpha < x_n \, \forall n \geq N \}
\]

We define \( \limsup x_n = \inf \mathcal{U} \) and \( \liminf x_n = \sup \mathcal{L} \).


2. Let \( \{x_n\}_{n=1}^{\infty} \) be any sequence.

(i) Show that \( \limsup x_n \neq -\infty \) if and only if \( \{x_n\}_{n=1}^{\infty} \) is bounded above. Find an example where \( x_n \) is bounded above and \( \limsup x_n = -\infty \). (Notice that the claim I made in class is not entirely correct)

(ii) Show that \( \liminf x_n \neq -\infty \) if and only if \( \{x_n\}_{n=1}^{\infty} \) is bounded below. Find an example where \( x_n \) is bounded below and \( \liminf x_n = \infty \). (Notice that the claim I made in class is not entirely correct)

3. Prove that if \( \{x_n\}_{n=1}^{\infty} \) converges, then \( \liminf x_n = \lim_{n \to \infty} x_n = \limsup x_n \).

4. Let \( \{x_n\}_{n=1}^{\infty} \) be any sequence that is bounded. Define a new sequence as follows :

\[
t_n = \sup\{x_m : m \geq n\} = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}.
\]

(i) Show that \( t_n \) is a monotone sequence.

(ii) Show that \( t_n \) is bounded.

(iii) Show that \( t_n \) converges and \( \lim_{n \to \infty} t_n = \limsup x_n \).

The above exercise justifies the name “\( \limsup \)”.

5. Let \( \{x_n\}_{n=1}^{\infty} \) be any sequence that is bounded. Define a new sequence as follows :

\[
s_n = \inf\{x_m : m \geq n\} = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}.
\]

(i) Show that \( s_n \) is a monotone sequence.

(ii) Show that \( s_n \) is bounded.

(iii) Show that \( s_n \) converges and \( \lim_{n \to \infty} s_n = \liminf x_n \).

The above exercise justifies the name “\( \liminf \)”.

6. Do 2.13.4, 2.13.5, 2.13.9, 2.13.10, 2.13.11 (for the last 3 problems, Exercises 4 and 5 could be useful).

7. Do 3.4.3, 3.4.4, 3.4.5, 3.4.6 (for this one you might need the result from the previous HW for problem 2.11.6 which showed that every subsequence of a convergent sequence converges), 3.4.11, 3.4.14, 3.4.15, 3.4.17, 3.4.21

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