HW 9 Solutions MAT 25

1. 2.12.2 Let $\{s_n\}$ be Cauchy.
Want to show: $\forall c \in \mathbb{R}$, $\{cs_n\}$ is Cauchy.

Pf: We know: $\forall \epsilon > 0, \exists N \in \mathbb{N}, |s_n - s_m| < \epsilon \ \forall n, m \geq N$

Consider any $c \in \mathbb{R}$.
Want to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, |s_n - s_m| < \gamma$
$\forall n, m \geq N$

Using $\epsilon = \dfrac{\gamma}{|c|}$, $\exists N \in \mathbb{N}$ s.t $|s_n - s_m| < \epsilon \ \forall n, m \geq N$

$$\Rightarrow |cs_n - cs_m| = |c||s_n - s_m|$$

$$< |c| \frac{\gamma}{|c|} = \gamma \quad \forall n, m \geq N.$$

■.

2.12.3 Let $\{s_n\}$, $\{t_n\}$ be Cauchy.
Thus: $\forall \epsilon > 0, \exists N \in \mathbb{N}, |s_n - s_m| < \epsilon \ \forall n, m \geq N$.
and $\forall \epsilon > 0, \exists N \in \mathbb{N}, |t_n - t_m| < \epsilon \ \forall n, m \geq N$.

Want to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, |(s_n + t_n) - (s_m + t_m)| < \epsilon$
$\forall n, m \geq N$

Using $\epsilon = \dfrac{\gamma}{2}$, $\exists N_1 \in \mathbb{N}$ $|s_n - s_m| < \dfrac{\gamma}{2} \ \forall n, m \geq N_1$

and $\exists N_2 \in \mathbb{N}$ $|t_n - t_m| < \dfrac{\gamma}{2} \ \forall n, m \geq N_2$

Set $N = \max\{N_1, N_2\}$.

Consider $n, m \geq N$

$$|(s_n+t_n)-(s_m+t_m)| = |(s_n-s_m)+(t_n-t_m)|$$
$$< |s_n-s_m| + |t_n-t_m|$$
$$< \frac{\gamma}{2} + \frac{\gamma}{2} \quad \left(\begin{array}{l}\text{since } n,m \geq N_1 \\ \text{and } n,m \geq N_2\end{array}\right.$$
$$= \gamma.$$

2.12.4 We know: $\forall \epsilon > 0, \exists N \in \mathbb{N}, |s_n - s_m| < \epsilon \ \forall n,m \geq N$.
Using $\epsilon = 1$, $\exists N \in \mathbb{N}$ s.t. $|s_n - s_m| < 1 \ \forall n,m \geq N$.

$\Rightarrow \forall m \geq N$,
$$|s_N - s_m| < 1.$$

$$\Rightarrow |s_m| - |s_N| \leq |s_N - s_m| < 1.$$

$$\Rightarrow |s_m| \leq |s_N| + 1. \quad (*)$$

Let $M = \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, |s_N|+1\}$.
Claim: $|s_n| \leq M \ \forall n \in \mathbb{N}$.

Pf: If $n < N$ then $|s_n| \leq \max\{|s_1|, \ldots, |s_{N-1}|\}$
$\leq M$.

If $n \geq M$. $|s_n| \leq |s_N| + 1$ by $(*)$.
$\leq M$.

2.12.5 Consider the series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Claim: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|s_{n+1} - s_n| < \epsilon \ \forall n \geq N$.

Pf: Given $\epsilon > 0$, let N be a natural number $\geq \frac{1}{\epsilon}$ (Archimedean Principle).

consider $n \geq N$.

$$|s_{n+1} - s_n| = \left| \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \right|$$

$$= \left| \frac{1}{n+1} \right| = \frac{1}{n+1}$$

$$< \frac{1}{n} \underset{\text{since } n \geq N}{\leq} \frac{1}{N} \underset{\text{since } N \geq \frac{1}{\epsilon}}{\leq} \epsilon$$

$\Rightarrow |s_{n+1} - s_n| < \epsilon$.

2.12.8. We know: $\forall \epsilon > 0, \exists N \in \mathbb{N}$, $|s_n - s_m| < \epsilon \ \forall n, m \geq N$.

Consider the subsequence $\{s_{n_k}\}$.

Want to prove: $\forall \gamma > 0, \exists K \in \mathbb{N}$ $|s_{n_k} - s_{n_j}| < \epsilon \ \forall k, j \geq K$

Consider $\gamma > 0$. Using Cauchy property of $\{s_n\}$ with $\epsilon = \gamma$, $\exists N \in \mathbb{N}$

s.t. $|s_n - s_m| < \gamma \ \forall n, m \geq N$.

Since $n_1 < n_2 < n_3 \ldots$ is a strictly increasing sequence, $\exists K \in \mathbb{N}$ s.t. $N \leq n_K$

Claim: $\forall k, j \geq K$, $|s_{n_k} - s_{n_j}| < \gamma$

since $k, j \geq K$.

$\Rightarrow n_k \geq n_K \geq N$

and $n_j \geq n_K \geq N$

$\Rightarrow |S_{n_k} - S_{n_j}| < \gamma$ since $n_k, n_j \geq N$

2. (i) First show $\limsup x_n \neq \infty \Rightarrow \{x_n\}_{n=1}^{\infty}$ is bounded above.

By def.

$\limsup x_n = \inf \mathcal{U}$. where $\mathcal{U} = \{\beta \in \mathbb{R} : \exists N \in \mathbb{N}, x_n < \beta \ \forall n \geq N\}$

since $\limsup x_n \neq \infty$

$\Rightarrow \mathcal{U} \neq \phi$.

Consider any $\beta \in \mathcal{U}$.

$\Rightarrow \underline{\underline{\exists N \in \mathbb{N}}}$ s.t. $x_n < \beta \ \forall n \geq N$.

Let $M = \max\{x_1, x_2, \ldots x_{N-1}, \beta\}$.

Claim: $x_n \leq M \ \forall n \in \mathbb{N}$, and thus $\{x_n\}_{n=1}^{\infty}$ is bounded above

Pf: Consider any $n \in \mathbb{N}$.

Case 1: $n < N$

$\Rightarrow x_n \leq \max\{x_1, x_2, \ldots, x_{N-1}\} \leq M$.

Case 2: $n \geq N$.

$\Rightarrow x_n < \beta \leq M$. ∎

since $n \geq N$

Next show that if $\{x_n\}_{n=1}^{\infty}$ is bounded above, then $\limsup x_n \neq \infty$.

Since $\{x_n\}_{n=1}^{\infty}$ is bounded above, $\exists M \in \mathbb{R}$ s.t.

$$x_n \leq M \quad \forall n \in \mathbb{N}.$$

$\Rightarrow$ M is an eventual upper bound : $N = 1$ works.

$\Rightarrow$ $M \in \mathcal{U}$

$\Rightarrow$ $\mathcal{U} \neq \phi$.

$\Rightarrow$ $\inf \mathcal{U} \neq \infty$. $\Rightarrow$ $\limsup x_n \neq \infty$. ∎

ii) This can be shown using similar arguments as Part (i).

~~3) We show first $\limsup x_n = \lim_{n\to\infty} x_n$~~

~~Let $L = \lim_{n\to\infty} x_n$~~

~~FACT 1: $\beta \in \mathcal{U} \Rightarrow L \leq \beta$.~~

~~Pf: [illegible].~~

~~[illegible]~~

~~Since $\beta \in \mathcal{U}$, $\exists N \in \mathbb{N}$ s.t. $x_n < \beta \ \forall n \geq N$.~~

3. We first show that $\limsup x_n = \lim_{n\to\infty} x_n$.
Let $L = \lim_{n\to\infty} x_n$

FACT 1 : Let $\beta \in \mathbb{R}$.
$L < \beta \implies \beta \in \mathcal{U}$.

Pf: Suppose $L < \beta$. $\therefore \beta - L > 0$.
Let $\epsilon = \beta - L > 0$ and use the def. of $\lim_{n\to\infty} x_n = L$

to get $N \in \mathbb{N}$ s.t. $|x_n - L| < \epsilon \quad \forall n \geq N$.

$\implies x_n - L \leq |x_n - L| < \beta - L \quad \forall n \geq N$
$\implies x_n < \beta \quad \forall n \geq N$.
Thus, $\beta \in \mathcal{U}$. $\blacksquare$

FACT 2 : $\beta \in \mathcal{U} \implies L \leq \beta$.
Pf: Assume $\beta \in \mathcal{U}$.
We ~~show~~ assume $L > \beta$ and arrive at a contradiction.
Since $\beta \in \mathcal{U}$, $\exists N_1 \in \mathbb{N}$ s.t. $x_n < \beta \quad \forall n \geq N_1$.
Since $L > \beta$, $\epsilon = \frac{L-\beta}{2} > 0$ and using the

def of $\lim_{n\to\infty} x_n = L$ we get $N_2 \in \mathbb{N}$ s.t.

$|x_n - L| < \epsilon \quad \forall n \geq N_2$.

Let $N = \max\{N_1, N_2\}$.

$\implies x_N < \beta$. since $N \geq N_1$

Also,

$$L - x_N \leq |x_N - L| < \epsilon = \frac{L-\beta}{2} \quad \text{since } N \geq N_2.$$

$$\Rightarrow \quad L - \frac{L-\beta}{2} < x_N$$

$$\Rightarrow \quad \frac{L+\beta}{2} < x_N$$

$$\Rightarrow \quad \frac{\beta+\beta}{2} < \frac{L+\beta}{2} < x_N \quad (\text{since } L > \beta).$$

$$\Rightarrow \quad \beta < x_N.$$

Thus, $x_N < \beta$ and $x_N > \beta$ which is a contradiction.

Therefore, $L \leq \beta$. $\boxtimes$

Now, from fact 2, L is a lower bound for $\mathcal{U}$.
$\Rightarrow \inf \mathcal{U} \geq L$.

__Claim__: $\inf \mathcal{U} \leq L$

Pf: ~~$\inf \mathcal{U} \geq L$~~ By contradiction.
Suppose $\inf \mathcal{U} > L$.

$$\Rightarrow \quad \frac{\inf \mathcal{U} + L}{2} > \frac{L+L}{2} = L.$$

since $\inf \mathcal{U} > L$.

Let $\beta = \dfrac{\inf \mathcal{U} + L}{2}$. Thus, $\beta > L$.

By Fact 2, $\beta \in \mathcal{U}$. Thus. $\beta \geq \inf \mathcal{U}$.

But. $\beta = \dfrac{\inf \mathcal{U} + L}{2} < \dfrac{\inf \mathcal{U} + \inf \mathcal{U}}{2} = \inf \mathcal{U}$.

Thus $\beta < \inf \mathcal{U}$ and $\beta \geq \inf \mathcal{U}$.
which is a contradiction.

Thus, $\inf \mathcal{U} \leq L$.

Therefore, $\inf \mathcal{U} = L$. ■.

The argument for $\liminf x_n = \lim_{n\to\infty} x_n$ is similar.
Prove the following facts:

FACT 1' : Let $\alpha \in \mathbb{R}$. $\alpha < L \Rightarrow \alpha \in \mathcal{L}$.
FACT 2' : $\alpha \in \mathcal{L} \Rightarrow \alpha \leq L$.
Use these 2 facts to show $\inf \mathcal{L} = L$.

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4. i) ~~Want to show~~ Claim: $\{t_n\}$ is non increasing.
Pf: Want to show $t_{n+1} \leq t_n$.

$$t_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\} \geq \sup\{x_{n+1}, x_{n+2}, \dots\} = t_{n+1}$$

The inequality since $A \subseteq B \Rightarrow \sup A \leq \sup B$.
and $\{x_{n+1}, x_{n+2}, \dots\} \subseteq \{x_n, x_{n+1}, x_{n+2}, \dots\}$.

ii) Since $\{x_n\}_{n=1}^{\infty}$ is bounded, $\exists M \in \mathbb{R}$ s.t.

$$x_n \leq M \quad \forall n \in \mathbb{N}.$$

$\Rightarrow t_n = \sup\{x_n, x_{n+1}, \dots\} \leq M$. since M is an upper bound.

$\Rightarrow t_n \leq M \quad \forall n \in \mathbb{N}$.

Need to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, |t_n - L| < \epsilon \ \forall n \geq N$.

1. Consider $\gamma > 0$.

Using $\epsilon = \gamma$ on the def. of $L = \inf \mathcal{U}$,

$$\exists \beta \in \mathcal{U} \quad \text{s.t.} \quad \beta < L + \gamma$$

$$\Rightarrow \exists N_1 \in \mathbb{N} \quad \text{s.t.} \quad x_n < \beta < L + \gamma \quad \forall n \geq N.$$

2. Set $N = N_1$.
3. Consider $n \geq N$.

$t_n \leq t_N$ since $\{t_n\}_{n=1}^{\infty}$ is non increasing

Also, $t_N = \sup\{x_N, x_{N+1}, x_{N+2} \dots\} \leq \beta$

since $\beta > x_n \quad \forall n \geq N$.

$$\Rightarrow t_n \leq t_N \leq \beta < L + \gamma.$$

Claim: $t_n \geq L$

Pf: Suppose not and $L > t_n$.

Let $\epsilon = L - t_n > 0$.

$$\Rightarrow t_n + \frac{\epsilon}{2} > t_n = \sup\{x_n, x_{n+1}, \dots\}.$$

$$\Rightarrow t_n + \frac{\epsilon}{2} > x_{n'} \quad \forall n' \geq N.$$

$$\Rightarrow t_n + \epsilon/2 \in \mathcal{U}.$$

And of course, since $\{t_n\}$ is non increasing
$t_n \leq t_1 \quad \forall n \in \mathbb{N}$.

Let $M' = \max\{|M|, |t_1|\}$.

$\Rightarrow -M' \leq -|t_1| \leq t_1 \leq t_n \leq M \leq |M| \leq M'$ $\forall n \in \mathbb{N}$.

$\Rightarrow -M' \leq t_n \leq M' \quad \forall n \in \mathbb{N}$.

$\Rightarrow |t_n| \leq M'$.

Thus. $\{t_n\}$ is bounded.

iii) By the Monotone Convergence Theorem,
$\{t_n\}$ is monotone and bounded, and therefore
$\{t_n\}$ converges.

~~Let~~ $\cancel{\lim_{n\to\infty} t_n}$.

Let $L = \limsup x_n = \inf \mathcal{U}$.

We now show $L = \lim_{n\to\infty} x_n$

~~Consider $\epsilon > 0$. Since $L = \inf \mathcal{U}$, $\exists \beta \in \mathcal{U}$.~~
~~s.t. $\beta < L + \epsilon$.~~
~~$\therefore \exists N \in \mathbb{N}$ s.t. $x_n < \beta = L + \epsilon \quad \forall n \geq N$.~~

~~Consider $n' \geq N$. Need to show $|t_n - L| < \epsilon$.~~
~~Since $n' \geq N$,~~

~~$t_{n'} = \sup\{x_{n'}, x_{n'+1}, \ldots\} \leq \beta$ since~~

$$\Rightarrow \quad t_n + \frac{\epsilon}{2} \geq L.$$

But $\quad t_n + \frac{\epsilon}{2} = t_n + \frac{L-t_n}{2} = \frac{L+t_n}{2}$

$$< \frac{L+L}{2} = L$$

since $t_n < L$.

Thus. $t_n + \frac{\epsilon}{2} \geq L$ and $t_n + \frac{\epsilon}{2} < L$ which is a contradiction.

Thus. $t_n \geq L$. ∎

Therefore $\quad t_n \geq L > L - \gamma.$

$$\Rightarrow L-\gamma < t_n < L+\gamma.$$

$$\Rightarrow \quad |t_n - L| < \gamma.$$

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5. Can be proved in similar manner as 4.

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6. $\limsup(-x_n) = \inf \mathcal{U}'$ where

$$\mathcal{U}' = \{\beta' \in \mathbb{R} : \exists N \in \mathbb{N},\ -x_n < \beta' \ \forall n \geq N\}.$$

Claim: $\mathcal{L} = -\mathcal{U}'$.

Pf: First show $\mathcal{L} \subseteq -\mathcal{U}'$

Consider $\alpha \in \mathcal{L}$.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $\alpha < x_n \ \forall n \geq N$.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $-x_n < -\alpha \quad \forall n \geq N$.

$\Rightarrow \quad -\alpha \in \mathcal{U}'$

$\Rightarrow \quad \alpha \in -\mathcal{U}'$

Next show $-\mathcal{U}' \subseteq \mathcal{L}$.

Consider $\beta \in -\mathcal{U}' \Rightarrow -\beta \in \mathcal{U}'$

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $-x_n < -\beta \quad \forall n \geq N$.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $x_n > \beta \quad \forall n \geq N$.

$\Rightarrow \quad \beta \in \mathcal{L}$.

Thus. $\mathcal{L} = -\mathcal{U}'$.

Therefore,

$$
\begin{aligned}
-\liminf x_n &= -\sup \mathcal{L}. \\
&= \inf -\mathcal{L}. \\
&= \inf \mathcal{U}' \\
&= \limsup(-x_n).
\end{aligned}
$$

2.13.5. Let $\mathcal{U}_1 = \{\beta \in \mathbb{R} : \exists N \in \mathbb{N} \; a_n < \beta \;\; \forall n \geq N\}$.
$\mathcal{U}_2 = \{\beta \in \mathbb{R} : \exists N \in \mathbb{N} \; b_n < \beta \;\; \forall n \geq N\}$.

Want to show $\inf \mathcal{U}_1 \leq \inf \mathcal{U}_2$.

Claim: $\mathcal{U}_2 \subseteq \mathcal{U}_1$

Pf: Consider any $\beta \in \mathcal{U}_2$.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $b_n < \beta \quad \forall n \geq N$

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $a_n \leq b_n < \beta \quad \forall n \geq N$.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $a_n < \beta \quad \forall n \geq N$.

$\Rightarrow \beta \in \mathcal{U}_1$.

From the claim, $\inf \mathcal{U}_1 \leq \inf \mathcal{U}_2$ follows.

A similar argument works for $\liminf a_n \leq \liminf b_n$.

2.13.9. Let $\mathcal{U}_1$ & $\mathcal{U}_2$ be as defined for 2.13.5.

Let $\mathcal{U}_3 = \{\beta \in \mathbb{R} : \exists N \in \mathbb{N} \; a_n + b_n < \beta \;\; \forall n \geq N\}$.
Let $\mathcal{U}_1 + \mathcal{U}_2 = \{\beta_1 + \beta_2 : \beta_1 \in \mathcal{U}_1, \beta_2 \in \mathcal{U}_2\}$.
Claim: $\mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}_3$.

Pf: Consider $\beta_1 \in \mathcal{U}_1$ & $\beta_2 \in \mathcal{U}_2$.
$\Rightarrow \exists N_1 \in \mathbb{N}$ s.t. $a_n < \beta_1 \quad \forall n \geq N_1$
and $\exists N_2 \in \mathbb{N}$ s.t. $b_n < \beta_n \quad \forall n \geq N_2$.

Let $N = \max\{N_1, N_2\}$.

Then $a_n + b_n < \beta_1 + \beta_2 \quad \forall n \geq N$.

$\Rightarrow \beta_1 + \beta_2 \in \mathcal{U}_3$.

Therefore $\limsup (a_n + b_n) = \inf \mathcal{U}_3$

$\leq \inf(\mathcal{U}_1 + \mathcal{U}_2)$

$= \inf \mathcal{U}_1 + \inf \mathcal{U}_2$

(using 1.6.14 from previous HW).

$= \limsup a_n + \limsup b_n$.

Example: $a_n = (-1)^n \qquad b_n = (-1)^{n+1}$

$\{a_n + b_n\} = \{0\}$.

$\Rightarrow \limsup (a_n + b_n) = 0$.

$\limsup a_n = 1$ and $\limsup b_n = 1$.

and $0 < 1 + 1 = 2$.

2.13.10. $\liminf (a_n+b_n) \geq \liminf a_n + \liminf b_n$.

Can be shown in a similar way
using $A \subseteq B \Rightarrow \sup A \leq \sup B$.
and $\sup(A+B) = \sup A + \sup B$
(1.6.13 from previous HW).

~~Alternate proof of 2.13.9:~~
~~$\limsup (a_n+b_n) = \lim_{n\to\infty} \sup$~~

2.13.11 define $t_n = \sup\{a_n, a_{n+1}, \ldots\}$.

$t'_n = \sup\{b_n, b_{n+1}, \ldots\}$.

$t''_n = \sup\{a_nb_n, a_{n+1}b_{n+1}, \ldots\}$.

Claim: $t''_n \leq t_nt'_n \quad \forall n \in \mathbb{N}$.

Pf: We show $t_nt'_n$ is an upper bound on $\{a_nb_n, a_{n+1}b_{n+1}, \ldots\}$.

Pick any $n' \geq n$.

$t_n \geq a_{n'} \qquad t'_n \geq b_{n'}$

$\Rightarrow t_nt'_n \geq a_{n'}b_{n'}$

$\Rightarrow t_nt'_n$ is an upper bound on $\{a_nb_n, a_{n+1}b_{n+1}, \ldots\}$

$\Rightarrow \quad t_n t'_n \geq \sup \{a_n b_n, a_{n+1} b_{n+1}, \ldots \}.$

by def. of sup.

$\Rightarrow \quad t_n t'_n \geq t''_n.$

Therefore. $\lim_{n\to\infty} t_n t'_n \geq \lim_{n\to\infty} t''_n$ (order property of limits).

$\Rightarrow \left(\lim_{n\to\infty} t_n\right)\left(\lim_{n\to\infty} t'_n\right) \geq \lim_{n\to\infty} t''_n$ (Algebra of limits).

$\Rightarrow \left(\limsup a_n\right)\left(\limsup b_n\right) \geq \limsup (a_n + b_n)$

7. 3.4.3 Claim: Either both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge

~~or~~ (e.g. $a_k = \frac{1}{2^k}$, $b_k = \frac{1}{2^k}$

$a_k + b_k = \frac{1}{2^{k-1}}$)

or both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverge

(e.g. $a_k = (-1)^k$, $b_k = (-1)^{k+1}$ $a_k + b_k = 0$.

Pf: We show that $\sum_{k=1}^{\infty} a_k$ converges iff $\sum_{k=1}^{\infty} b_k$ converges.

This will prove the claim.

If $\sum a_k$ converges., then $\sum(a_k+b_k) - \sum a_k$
~~converges~~ $= \sum(a_k+b_k-a_k)$
$= \sum b_k$
converges.

Similarly, if $\sum b_k$ converges, then $\sum(a_k+b_k) - \sum b_k$
$= \sum(a_k+b_k-b_k)$
$= \sum a_k$
converges.

3.4.4. We can say that both $\sum a_k$ & $\sum b_k$ cannot converge, as this would contradict Theorem 3.8.

Example where both diverge: $a_k=1$, $b_k=1$.

Example where $\sum a_k$ diverges, $\sum b_k$ converges:
$a_k=1$, $b_k=0$.

" " $\sum a_k$ converges, $\sum b_k$ diverges:
$a_k=0$, $b_k=1$.

3.4.5. $\sum_{k=1}^{\infty} a_k$ may converge or diverge.

Example where $\sum_{k=1}^{\infty} a_k$ converges: $a_k = 0$. $\Rightarrow \sum a_k = 0$.

$$\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1}) = \sum_{k=1}^{\infty} 0 = 0.$$

Example where $\sum_{k=1}^{\infty} a_k$ diverges: $a_k = (-1)^k \Rightarrow \sum a_k$ diverges

$$\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1}) = \sum_{k=1}^{\infty}\left((-1)^{2k} + (-1)^{2k-1}\right)$$

$$= \sum_{k=1}^{\infty} 0$$

$$= 0.$$

3.4.6. $\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow \sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})$ converges.

Pf: Let $s_n = \sum_{k=1}^{n} a_k$ be the partial sums for $\sum_{k=1}^{\infty} a_k$

Let $s'_n = \sum_{k=1}^{n}(a_{2k} + a_{2k-1})$

$$= (a_1 + a_2) + (a_3 + a_4) + \ldots + (a_{2n-1} + a_{2n})$$

$$= s_{2n}.$$

Therefore, $\{s'_n\}_{n=1}^{\infty}$ is a subsequence of $\{s_n\}_{n=1}^{\infty}$

Since $\{s_n\}$ converges, the subsequence $\{s'_n\}_{n=1}^{\infty}$ also converges (by 2.11.6 (i)). Thus, $\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})$ converges.

3.4.11 $\quad 0.1234512345\ldots$

$$= 12345\times10^{-5} + 12345\times10^{-10} + 12345\times10^{-15} + \ldots$$

$$= 12345\times10^{-5}\left(1 + 10^{-5} + 10^{-10} + \ldots\right)$$

$$= 12345\times10^{-5}\left(1 + r + r^2 + \ldots\right)$$

where $r = 10^{-5} < 1.$

Using the formula $1 + r + r^2 + \ldots = \dfrac{1}{1-r}.$

$$= 12345\times10^{-5}\times\frac{1}{1-10^{-5}}$$ (We also use Theorem 3.9 here)

$$= \frac{12345}{10^5-1}$$

$$= \frac{12345}{99999}$$

3.4.14. We add $\dfrac{1}{4} + \dfrac{1}{4}\times\dfrac{1}{4} + \dfrac{1}{4}\times\dfrac{1}{4}\times\dfrac{1}{4} + \ldots$

$$= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$$

$$= \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\right)$$

$$= \frac{1}{4}\left(1 + r + r^2 + \ldots\right)$$

where $r = 1/4 < 1.$

$$= \frac{1}{4}\times\frac{1}{1-1/4} = \frac{1}{3}$$

3.4.15 $\displaystyle\sum_{k=1}^{\infty} \log\left(\frac{k+1}{k}\right)$

Partial sum : $\displaystyle S_n = \sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right)$

$$= \sum_{k=1}^{n} \log(k+1) - \log(k)$$

$$= (\log 2 - \log 1) + (\log 3 - \log 2) + (\log 4 - \log 3) + \dots$$
$$+ \dots (\log(n+1) - \log n)$$

[Telescopic sum].

$$= \log(n+1) - \log 1 = \log(n+1).$$

Claim : $\{S_n\} = \{\log(n+1)\}$ diverges to ∞.

Pf: Need to show $\forall\, M \in \mathbb{R}$, $\exists\, N \in \mathbb{R}$ s.t. $S_n \geq M \;\forall n \geq N$.

If $M \leq 0$, Let $N = 1$.

For $n \geq N$, $S_n = \log(n+1) \geq \log(N+1) = \log 2 > 0. \geq M.$

If $M > 0$, Let $N \geq e^M$ [Archimedean principle].

For $n \geq N$,

$$S_n = \log(n+1) \geq \log n \geq \log N \geq \log e^M = M.$$

Thus in both cases, we are good. $\blacksquare$

Thus, $\sum_{k=1}^{\infty} \log\left(\frac{k+1}{k}\right)$ diverges.

<u>3.4.17</u> $2 + \frac{2}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \dots$

$= 2\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} + \dots\right.$

$= 2\left(1 + \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^4 + \dots\right)$

$= 2(1 + r + r^2 + r^3 + \dots)$

where $r = \frac{1}{\sqrt{2}} < 1$.

$= 2 \times \frac{1}{1 - 1/\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}-1}$.

<u>3.4.21</u> We have $a > 0$, $b > 0$.

<u>Claim</u>: $\sum_{k=1}^{\infty} \frac{1}{a+kb}$ diverges.

$s_n = \sum_{k=1}^{n} \frac{1}{a+kb} \geq \sum_{k=1}^{n} \frac{1}{ka+kb} = \frac{1}{(a+b)}\left(\sum_{k=1}^{n} \frac{1}{k}\right)$

since $a \leq ka \quad \forall k \geq 1$.

~~Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to ∞~~

Since $s'_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges to ∞.

$\left(\frac{1}{a+b}\right) s'_n = \frac{1}{a+b}\left(\sum_{k=1}^{n} \frac{1}{k}\right)$ also diverges to ∞.

since $s_n \geq \frac{1}{a+b} s'_n$

and $s'_n \to \infty$.

we have $s_n \to \infty$.

Thus, $\sum_{k=1}^{\infty} \frac{1}{a+kb}$ diverges.