# HW 8 Solutions MAT 25

1. 2.7.3

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{1}{2+1/n^2}$$

We know $\lim 1/n^2 = 0$.

$$\Rightarrow \lim 2 + \frac{1}{n^2} = \lim 2 + \lim \frac{1}{n^2} = 2.$$

(Theorem 2.15

$$\therefore \lim \frac{1}{2+1/n^2} = \frac{\lim 1}{\lim 2 + 1/n^2} = \frac{1}{2}$$

(Theorem 2.17).

2.7.8. They don't help since $\{\sqrt{n+1}\}$ and $\{\sqrt{n}\}$ are divergent sequences.

But we can do the following:

$$\lim\left(\sqrt{n+1}-\sqrt{n}\right) = \lim\left(\sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$$

$$= \lim \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim \frac{1/\sqrt{n}}{\sqrt{1+1/n}+1}$$

Now $\lim \frac{1}{\sqrt{n}} = 0$ and $\lim \frac{1}{n} = 0$.

$$\Rightarrow \lim \left(1+\frac{1}{n}\right) = \lim 1 + \lim \frac{1}{n} = 1$$

$$\Rightarrow \lim \sqrt{1+\frac{1}{n}} = \sqrt{1} = 1.$$

$$\Rightarrow \lim \left(\sqrt{1+\frac{1}{n}} + 1\right) = 1 + 1 = 2.$$

$$\Rightarrow \lim \frac{1/\sqrt{n}}{\sqrt{1+1/n}+1} = \frac{\lim 1/\sqrt{n}}{\lim\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{0}{2} = 0$$

Similary, $\lim \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

$$= \lim \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{\lim 1}{\lim\left(\sqrt{1+\frac{1}{n}}\right) + \lim 1}$$

$$= \frac{1}{2}$$

2. Consider the sequences $a_n = \alpha$, $\beta_n = \beta$.

$\Rightarrow \quad a_n \le s_n \quad \forall\ n \in \mathbb{N}.$

$\Rightarrow \quad \lim a_n \le \lim s_n$ (Order property of limits)

$\Rightarrow \quad \alpha \le \lim s_n.$

Similarly,

$s_n \le b_n$

$\Rightarrow \lim s_n \le \lim b_n$ (Order property of limits).

$\Rightarrow \lim s_n \le \beta.$

3. 2.8.1.

Let $s_n = -\dfrac{1}{n}$, $t_n = \dfrac{1}{n}$.

Then $s_n < t_n$

But $\lim s_n = 0 = \lim t_n$.

2.8.2. Define $a_n = \dfrac{a}{n}$ $\quad b_n = \dfrac{b}{n}$.

Then since $\dfrac{1}{n} \to 0$.

By Theorem 2.14, $a_n = a\left(\dfrac{1}{n}\right) \longrightarrow a \cdot 0 = 0$

and $b_n = b\left(\dfrac{1}{n}\right) \longrightarrow b \cdot 0 = 0$

Also since $s_n \in [a, b]$

$\Rightarrow a \le s_n \le b$

$$\Rightarrow \quad \frac{a}{n} \le \frac{s_n}{n} \le \frac{b}{n}.$$

$$\Rightarrow \quad a_n \le s_n \le b_n$$

We know $\lim a_n = \lim b_n = 0$.

By the squeeze theorem, $s_n$ converges, and $\lim s_n = 0$.

4. Yes, $\lim s_n \le \lim t_n$.
Let $\lim s_n = S$ and $\lim t_n = T$.

Pf: Suppose for contradiction,
$$T < S.$$

Let $\epsilon = \frac{S-T}{2} > 0$.

By def. of $\lim s_n = S$, $\exists N_1 \in \mathbb{N}$ s.t.
$$|s_n - S| < \epsilon \quad \forall n \ge N_1.$$
By def of $\lim t_n = T$, $\exists N_2 \in \mathbb{N}$ s.t.
$$|t_n - T| < \epsilon \quad \forall n \ge N_2.$$

$$\therefore \quad S - s_n < \epsilon = \frac{S-T}{2}$$

$$\Rightarrow S - \frac{S-T}{2} < s_n \quad \forall n \ge N_1$$

$$\Rightarrow \frac{S+T}{2} < s_n \quad \forall n \ge N_1$$

Similarly $t_n - T < \epsilon = \frac{S-T}{2}$

$$\Rightarrow t_n < \frac{S+T}{2} \quad \forall n \ge N_2$$

Let $N' = \max\{N_1, N_2, N\}$.

~~Now~~

$\Rightarrow \quad t_{N'} < \frac{S+T}{2} < s_{N'}$

from $N' \geq N_2$ from $N' \geq N_1$

But $s_{N'} \leq t_{N'}$ since $N' \geq N$.

This is a contradiction.

5. Modify the proof of the squeeze theorem as in Problem 4.

6. 2.8.4 Want to show: $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, t_n \geq M \ \forall n \geq N$.

We know: $\forall L \in \mathbb{R}, \exists N \in \mathbb{N}, s_n \geq L \ \forall n \geq N$.

1. Consider $M \in \mathbb{R}$. 2. Using $L = M$, we get $N_1 \in \mathbb{N}$ st. $s_n \geq L \ \forall n \geq N_1$.

3. Let $N = N_1$.

4. Consider $n \geq N$

$t_n \geq s_n \geq L$ since $n \geq N = N_1$

$\blacksquare$.

7. We know: $\forall L \in \mathbb{R}, \exists N \in \mathbb{N}, \; s_n \geq L \; \forall n \geq N$.
Want to show: $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, \; t_n \geq M \; \forall n \geq N$.

1. Consider $M \in \mathbb{R}$.
2. Using $L = M$, $\exists N_1 \in \mathbb{N}$ s.t. $s_n \geq L \; \forall n \geq N_1$.
3. Let $N_2 = \max\{N_1, N\}$.
4. Consider $n \geq N_2$.
5. $t_n \geq s_n$ (since $n \geq N_2 \geq N$).

   and $s_n \geq L = M$ (since $n \geq N_2 \geq N_1$).

   $\Rightarrow \underline{t_n \geq M}$. $\blacksquare$

8. Let $\lim \frac{s_{n+1}}{s_n} = L < 1$.

i) Set $\beta = \frac{1+L}{2}$.

Clearly $\beta = \frac{1+L}{2} < \frac{1+1}{2} = 1$ since $L<1$.

Now using $\epsilon = \frac{1-L}{2} > 0$, in the def of

$\lim \frac{s_{n+1}}{s_n} = L$ we get $N \in \mathbb{N}$

s.t. $\left| \frac{s_{n+1}}{s_n} - L \right| < \frac{1-L}{2} \quad \forall n \geq N$

$$\Rightarrow \quad \frac{s_{n+1}}{s_n} - L < \frac{1-L}{2} \quad \forall\, n \geq N.$$

$$\Rightarrow \quad \frac{s_{n+1}}{s_n} < L + \frac{1-L}{2} = \frac{1+L}{2} = \beta \quad \forall\, n \geq N.$$

$$\Rightarrow \quad \frac{s_{n+1}}{s_n} < \beta \quad \forall\, n \geq N.$$

Therefore $\beta = \frac{1+L}{2}$ and the above $N$ works.

ii) ~~Claim~~ Let $C = s_N$ where $N$ is the number from part (i).

<u>Claim</u>: $s_n \leq C\beta^{n-N} \quad \forall\, n \geq N.$

<u>Pf</u>: By induction: Base case <u>$n = N$</u>

$$s_N = C \leq C\beta^{N-N} = C.$$

<u>Induction step</u>: Assume $s_{n-1} \leq C\beta^{(n-1)-N}$

we know $\frac{s_n}{s_{n-1}} \leq \beta \quad \forall\, n \geq N+1$

$$\Rightarrow s_n \leq \beta s_{n-1}$$

$$\Rightarrow s_n \leq \beta\left(C\beta^{(n-1)-N}\right)$$

$$= C\beta^{n-N}$$

∎

iii) Define two sequences:

$$a_n = 0 \qquad b_n = \frac{C}{\beta^N}\beta^n$$

$\lim a_n = 0.$

By Problem 1 in HW 6, $\lim \beta^n = 0.$

$$\Rightarrow \lim b_n = \lim \frac{C}{\beta^N}\beta^n$$

$$= \frac{C}{\beta^N}\lim \beta^n = \frac{C}{\beta^N}\cdot 0$$

$$= 0.$$

Also. $a_n < s_n < b_n \quad \forall n \geq N.$

$s_n$ are strictly positive (for $a_n < s_n$); part (ii) (for $s_n < b_n$).

Using Problem 5, $\{s_n\}$ converges and $\lim s_n = 0.$ $\boxtimes$

9. Parts (i) + (ii) are similar to 8(i) + (ii).

iii) Need to show: $\forall M \in \mathbb{R}$, $\exists N_1 \in \mathbb{N}$ s.t. $s_n \geq M$ $\forall n \geq N_1$

1. Consider $M \in \mathbb{R}$. 2. Choose $N_2 \geq \log_\beta\left(\frac{M\beta^N}{C}\right)$

where $N$ is from part (i). (Archimedean principle)

2. Let $N_1 = \max\{N, N_2\}$.

3. Consider $n \geq N_1$.

$s_n \geq C\beta^{n-N}$ (since $n \geq N$).

$$\Rightarrow \quad s_n \geq \frac{C}{\beta^N}\beta^n$$

$$\geq \frac{C}{\beta^N}\beta^{N_2} \qquad \left(\text{since } \beta > 1 \text{ and } n \geq N_2\right)$$

$$\geq \frac{C}{\beta^N}\beta^{\log_\beta\left(\frac{M\beta^N}{C}\right)} \qquad \left(\text{since } N_2 \geq \log_\beta\left(\frac{M\beta^N}{C}\right)\right)$$

$$= \frac{C}{\beta^N}\cdot\frac{M\beta^N}{C}$$

$$= M. \qquad \blacksquare$$

10. The sequence may diverge or converge — we cannot say.

Example where sequence converges: $s_n = 1$. $\frac{s_{n+1}}{s_n} = 1$.

" " " diverges: $s_n = n$ : $\frac{s_{n+1}}{n} = \frac{n+1}{n}$

$$\Rightarrow \lim \frac{s_{n+1}}{n} = \lim \frac{n+1}{n}$$

$$= \lim 1 + \frac{1}{n}$$

$$= 1.$$

11. By Problem 12 in HW 6,

$s_{n+1} \geq s_n$.

Thus $s_n$ is nondecreasing.

Also, $s_n \leq 2 \quad \forall n \in \mathbb{N}$.

Thus $\{s_n\}$ is bounded.
By the Monotone Convergence Theorem,
$\{s_n\}$ converges.

The limit is 2. Here is the logic:
Let $\lim s_n = s$.
Let $a_n = s_{n+1}$ be a new sequence.

Clearly $\lim a_n = \lim s_n = s$.

We know $a_n = \sqrt{2+s_n} \quad \forall n \in \mathbb{N}$.

$$\Rightarrow a_n^2 = 2 + s_n$$

$$\Rightarrow \lim a_n^2 = \lim (2+s_n)$$

$$\Rightarrow (\lim a_n)^2 = 2 + \lim s_n$$ (using Algebra of limits)

$$\Rightarrow s^2 = 2 + s$$

$$\Rightarrow s^2 - s - 2 = 0$$

$$\Rightarrow (s-1)(s-2) = 0$$

$$\Rightarrow s = 1 \text{ or } s = 2.$$

$s$ cannot be 1 since $s_1 > 1$ and $\{s_n\}$ is nondecreasing.

$$\Rightarrow s = 2.$$

12. By Induction :

Base case : $n=1$

$$S_{2^1-1} = S_1 = 1$$

$$2 - \frac{1}{2^{1-1}} = 2-1 = 1.$$

Thus LHS = RHS ✓.

Induction Step : Assume $S_{2^{n-1}-1} \le 2 - \frac{1}{2^{n-2}}$

$$S_{2^n-1} = \underbrace{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(2^{n-1}-1)^2}}_{= S_{2^{n-1}-1}} + \frac{1}{(2^{n-1})^2} + \cdots + \frac{1}{(2^n-1)^2}$$

$$= S_{2^{n-1}-1} + \underbrace{\frac{1}{(2^{n-1})^2} + \frac{1}{(2^{n-1}+1)^2} + \cdots + \frac{1}{(2^n-1)^2}}_{2^{n-1} \text{ terms}}$$

$$\le S_{2^{n-1}-1} + \frac{1}{(2^{n-1})^2} + \frac{1}{(2^{n-1})^2} + \cdots + \frac{1}{(2^{n-1})^2}$$

$$= S_{2^{n-1}-1} + \frac{2^{n-1}}{(2^{n-1})^2} = S_{2^{n-1}-1} + \frac{1}{2^{n-1}}$$

by I.H. $\nearrow$ $\le 2 - \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$ ∎

Since $\dfrac{1}{(n+1)^2} \geq 0$

$S_{n+1} \geq S_n$ and so $\{S_n\}$ is nondecreasing.

Also for any $n \in \mathbb{N}$.

$$S_n \leq S_{2^n-1}$$ (since $n \leq 2^n - 1$ and $\{S_n\}$ is nondecreasing)

$$\Rightarrow S_n \leq 2 - \frac{1}{2^{n-1}} < 2$$

Thus $\{S_n\}$ is bounded above by 2.

By the monotone convergence theorem, $\{S_n\}$ converges.

B. 2.9.2.

$$t_n = \sqrt{t_{n-1} + 1}$$

<u>Claim 1</u>: $t_n$ is nondecreasing, i.e., $t_{n+1} \geq t_n$

Proof by induction: Base case: $n=1$

$t_2 = \sqrt{2} > t_1 = 1$.

Induction step: assume $t_n \geq t_{n-1}$

$$\Rightarrow t_n + 1 \geq t_{n-1} + 1$$

$$\Rightarrow \sqrt{t_n + 1} \geq \sqrt{t_{n-1} + 1}$$

$$\Rightarrow t_{n+1} \geq t_n$$

Claim 2: $t_n < 3 \quad \forall n \in \mathbb{N}$.

Proof by induction:

Base case: $t_1 = 1 < 3$ ✓.

Induction Step: Assume $t_{n-1} < 3$

$$\Rightarrow t_{n-1} + 1 < 3 + 1$$

$$\Rightarrow \sqrt{t_{n-1}+1} < \sqrt{4} = 2 < 3.$$

$$\Rightarrow t_n < 3.$$

Thus, $\{t_n\}$ is nondecreasing + bounded above and therefore converges.

Let $\lim t_n = T$.

Using the same analysis as Problem 11

$$T^2 = T + 1$$

$$\Rightarrow T^2 - T - 1 = 0.$$

$$\Rightarrow T = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad T = \frac{1-\sqrt{5}}{2}$$

Since $t_1 = 1$ and $\{t_n\}$ is nondecreasing, and $\frac{1-\sqrt{5}}{2} < 0$.

$$T = \frac{1+\sqrt{5}}{2} \quad \text{(a.k.a. the golden ratio)}.$$

2.9.4.

$$t_n = t_{n-1}\frac{(2n-1)}{2n}$$

Since $\dfrac{2n-1}{2n} < 1 \quad \forall n \in \mathbb{N}$.

$\Rightarrow \quad t_n < t_{n-1} \quad \forall n \in \mathbb{N}$.

Thus $\{t_n\}$ is a decreasing sequence.

Also, clearly $t_n \geq 0 \quad \forall n \in \mathbb{N}$.

Thus, $\{t_n\}$ is decreasing + bounded below.

$\Rightarrow$ $t_n$ converges by the monotone convergence theorem.

2.9.5.

$$t_{n+1} = \frac{2\cdot4\cdot6\cdot\ldots\cdot(2n)(2n+2)}{1\cdot3\cdot5\cdot\ldots(2n-1)(2n+1)\ (n+1)^2}$$

$$= t_n\ \frac{(2n+2)}{(2n+1)(n+1)}$$

~~Since~~ Claim: $\dfrac{2n+2}{(2n+1)(n+1)} < 1 \quad \forall n \in \mathbb{N}$.

Consider $(2n+1)(n+1) - (2n+2)$

$= 2n^2 + 3n + 1 - 2n - 2$

$= 2n^2 + n - 1$

$= n(2n+1) - 1$

Since $n \geq 1$ and $2n+1 > 1 \quad \forall n \in \mathbb{N}$.

$$n(2n+1) > 1$$

$$\Rightarrow n(2n+1) - 1 > 0$$

$$\Rightarrow (2n+1)(n+1) - (2n+2) > 0.$$

$$\Rightarrow \frac{2n+2}{(2n+1)(n+1)} < 1.$$

$$\therefore \quad t_{n+1} < t_n \quad \forall\, n \in \mathbb{N}.$$

So $\{t_n\}$ is ~~...~~ decreasing.
Also clearly $t_n \geq 0 \quad \forall\, n \in \mathbb{N}$.

Thus $\{t_n\}$ converges by the Monotone Convergence Theorem.

Q14. 2.11.1.
By letting $n_i = i$ in the def. of subsequence we find the original sequence is a subsequence.

2.11.2. Suppose $\{s_{n_i}\}$ be a subsequence.

Let $a_i = s_{n_i}$ $\forall i \in \mathbb{N}$. be another sequence with

~~indices in~~ So a subsequence of $\{s_{n_i}\} = \{a_i\}$

is given by a sequence of natural numbers.

$m_1 < m_2 < m_3 \ldots$

$\{a_{m_j}\}$ forms a subsequence of $\{a_i\} = \{S_{n_j}\}$.

Define the sequence of natural numbers.

$\{n_{m_j}\}$

$n_{m_1}, n_{m_2}, n_{m_3}, \ldots$

since $m_1 < m_2 < m_3 \ldots$

$n_{m_1} < n_{m_2} < n_{m_3} < \ldots$

Thus $\{S_{n_{m_j}}\}$ is a subsequence of $\{S_n\}$.

But $S_{n_{m_j}} = a_{m_j}$ by definition of $\{a_i\}$.

Thus the subsequence $\{a_{m_j}\}$ of $\{S_{n_j}\}$.

is the sequence $\{S_{n_{m_j}}\}$ which is a subsequence of $\{S_n\}$.

**2.11.3** No.

Counterexample : $s_n = (-1)^n = t_n$.

Let $\begin{pmatrix} s_{n_k} = -1 \\ \forall k \in \mathbb{N} \end{pmatrix}$ which is a subsequence of $s_n$

Let $\begin{pmatrix} t_{m_k} = 1 \\ \forall k \in \mathbb{N} \end{pmatrix}$ which is a subsequence of $t_n$.

$\{s_{n_k} + t_{m_k}\}_{k=1}^{\infty} = \{0\}_{k=1}^{\infty}$ is NOT a subsequence

of $\{s_n + t_n\}_{n=1}^{\infty} = \{(-1)^n \cdot 2\}_{n=1}^{\infty}$

**2.11.6.** a) True

Clearly if all subsequences converge, then the sequence converges because the sequence is a subsequence of itself.

To show the other direction:

$\{s_n\}$ converges $\Rightarrow$ $\{s_{n_k}\}$ converges.

~~1. Consider $\epsilon > 0$.~~

~~Need to~~ ~~2. By def. of $\{s_n\}$ converges~~

Let $s_n \rightarrow L$.

Claim : $s_{n_k} \rightarrow L$.

1. Consider $\epsilon > 0$.
2. By def. of $s_n \rightarrow L$, $\exists N \in \mathbb{N}$. s.t. $|s_n - L| < \epsilon \quad \forall n \geq N$.

Since $n_1 < n_2 < n_3 < \ldots$

is an increasing sequence of natural numbers,

$\exists\ \hat{k} \in \mathbb{N}$ s.t. $n_{\hat{k}} \geq N$.

$\Rightarrow$ $n_k \geq n_{\hat{k}} \geq N \quad \forall\ k \geq \hat{k}$.

$\Rightarrow$ $|S_{n_k} - L| < \epsilon \quad \forall\ k \geq \hat{k}$

since $n_k \geq N$ and $|S_n - L| < \epsilon$ $\forall\ n \geq N$

b) True. ~~Very~~ ~~similar~~ Simply use the fact that a subsequence is a subset of the original sequence.

c) True since $\{S_n\}$ is monotonic.

$\Rightarrow$ $S_n \leq S_m\ \forall n \geq m$ or $S_n \geq S_m\ \forall n \geq m$.

$\Rightarrow$ $S_{n_k} \leq S_{n_j}\ \forall\ k \geq j$ (since $n_k \geq n_j$) or $S_{n_k} \geq S_{n_j}\ \forall k \geq j$ (since $n_k \geq n_j$).

d) False. If all subsequences are divergent, the original sequence is also divergent. (since the sequence is a subsequence of itself).

However: $S_n = (-1)^n$ has convergent subsequence.

2.11.18. Every unbounded sequence is either unbounded above or unbounded below (or both):

i.e., either $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$, s.t. $s_N \geq M$.

or $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $s_N \leq M$.

Case 1: $\{s_n\}$ is unbounded above:

Since $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $s_N \geq M$.

we construct a subsequence as follows:

$s_{n_1} = s_1$

Letting $M = s_{n_1} + 1$, $\exists n_2 \in \mathbb{N}$ s.t. $s_{n_2} \geq s_{n_1} + 1$

Letting $M = s_{n_2} + 1$, $\exists n_3 \in \mathbb{N}$ s.t. $s_{n_3} \geq s_{n_2} + 1$

Defining the subsequence thus, we find a strictly increasing subsequence.

Case 2: $\{s_n\}$ is unbounded below:

using a similar analysis one can find a strictly decreasing subsequence.

2.11.20. Suppose for the sake of contradiction that $\{x_n\}$ does not converge to $L$.

$\therefore$ ~~$\forall \epsilon > 0, \exists$~~ $\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N$ s.t. $|s_n - L| \geq \epsilon$.

Consider the following subsequence:

For $N=1$, $\exists\, n_1 \geq N$ s.t. $|S_{n_1} - L| \geq \epsilon$.

For $N = n_1$, $\exists\, n_2 \geq n_2$ s.t. $|S_{n_2} - L| \geq \epsilon$.

For $N = n_2$, $\exists\, n_3 \geq n_2$ s.t. $|S_{n_3} - L| \geq \epsilon$.

$\vdots$

Continuing thus, we find a subsequence $\{S_{n_i}\}$ such that $|S_{n_i} - L| \geq \epsilon \quad \forall\, i \in \mathbb{N}$.

Clearly $\{S_{n_i}\}$ does not converge to $L$ since

$$|S_{n_i} - L| \geq \epsilon \quad \forall\, i \in \mathbb{N}.$$

Now we know that ~~every~~ $\{S_{n_i}\}$ has a further subsequence converging to $L$.

But this is not possible since $|S_{n_i} - L| \geq \epsilon \ \forall i \in \mathbb{N}$.

This is a contradiction to the hypothesis.

2.11.21 Since $x = \sup\{x_n : n \in \mathbb{N}\}$.
$\forall \epsilon > 0$, $\exists n \in \mathbb{N}$, s.t. $x_n > x - \epsilon$.

Using $\epsilon = 1$, $\exists n_1 \in \mathbb{N}$ s.t. $x_{n_1} > x - 1$.

Since $x_{n_1} < x$, $x - x_{n_1} > 0$.
Using $\epsilon = \min\{\frac{1}{2}, x - x_{n_1}\}$, $\exists n_2 \in \mathbb{N}$ s.t.

$$x_{n_2} > x - \epsilon \geqslant x - (x - x_{n_1}) = x_{n_1}$$

Define $x_{n_j}$ recursively

Using $\epsilon = \min\{\frac{1}{j}, x - x_{n_{j-1}}\}$, $\exists n_j \in \mathbb{N}$ s.t.

$$x_{n_j} > x - \epsilon \geqslant x - (x - x_{n_{j-1}}) = x_{n_{j-1}}$$

Thus $\{x_{n_j}\}$ is an increasing sequence.

Also given any $\gamma > 0$, consider $J \geqslant \frac{1}{\gamma}$
For all $j \geqslant J$
We know $x_{n_j} > x - \frac{1}{j}$ (since we chose $\epsilon \leq \frac{1}{j}$ at the $j$th stage).

$$\Rightarrow |x - x_{n_j}| < \frac{1}{j} < \gamma \quad \forall j \geqslant J.$$

Thus, $x_{n_j} \longrightarrow x$.