HW 7 Solutions — MAT 25

1. **2.5.2.**

Need to show: $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, s_n \geq M \; \forall n \geq N$.

1. Consider arbitrary $M \in \mathbb{R}$.
2. ~~Let N be a natural no. ≥ √M~~

   If $M < 0$, let $N = 1$.

   Else, let N be a natural number $\geq \sqrt{M}$ (Archimedean principle).
3. Consider $n \geq N$.

   If $M < 0$, $s_n = n^2 \geq 0 > M$.

   If $M > 0$, $s_n = n^2 \geq N^2 \geq M$ ∎

2.5.3

1. Consider $M \in \mathbb{R}$.
2. Let N be a natural number $\geq M+1$ (Archimedean principle).
3. Consider $n \geq N$

$$s_n = \frac{n^3+1}{n^2+1} = \frac{n^3+n-n+1}{n^2+1} = n - \frac{n}{n^2+1} + \frac{1}{n+1}$$

$$\geq n - \frac{n}{n^2+1} \quad \left(\text{since } \frac{1}{n+1} \geq 0\right)$$

$$\geq n-1 \quad \left(\text{since } \frac{n}{n^2+1} \leq 1\right)$$

$$\geq N-1 \quad (\text{since } n \geq N)$$

$$\geq M+1-1 \quad (\text{since } N \geq M+1)$$

$$= M.$$

∎

2.5.5. We know : $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, \; s_n \geq M \;\; \forall n \geq N$

Want to show : $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, \; s_n^2 \geq M \;\; \forall n \geq N$.

1. Consider arbitrary $M \in \mathbb{R}$.
2. If $M < 0$, set $N = 1$.

   Else let $M' = \sqrt{M} \geq 0$.

   and get $N' \in \mathbb{N}$ s.t. $s_n \geq M' \;\; \forall n \geq N'$

   using def. of $s_n \to \infty$.

   Set $N = N'$
3. Consider $n \geq N$.

If $M \geq 0$. since $n \geq N = N'$

$s_n \geq M' = \sqrt{M}$

thus $s_n \geq 0$ and $\sqrt{M} \geq 0$.

$\Rightarrow s_n^2 \geq (\sqrt{M})^2 = M$ $\blacksquare$.

If $M < 0$, $(s_n)^2 > 0 > M$ $\blacksquare$.

2. We know : $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, \; s_n \geq M \;\; \forall n \geq N$.

Need to show : $\forall \epsilon > 0, \exists N \in \mathbb{N}, \; \left|\frac{1}{s_n} - 0\right| < \epsilon \;\; \forall n \geq N$.

1. Consider $\epsilon > 0$.
2. Let $M = \frac{2}{\epsilon} > 0$. Get $N' \in \mathbb{N}$ s.t. $s_n \geq M \;\; \forall n \geq N'$

   using definition of $s_n \to \infty$.

   Set $N = N'$.
3. Consider $n \geq N$.

$\Rightarrow s_n \geq M > 0$.

$\Rightarrow \left|\frac{1}{s_n}\right| = \frac{1}{s_n} \leq M = \frac{\epsilon}{2} < \epsilon$. $\blacksquare$.

The converse is not true: $s_n = (-1)^n n$.

$$\frac{1}{s_n} = \frac{(-1)^n}{n} \longrightarrow 0. \text{ but}$$

$$s_n \not\to \infty.$$

3. <u>2.5.6</u> From the previous exercise $\frac{1}{x_n} \to 0$.

<u>Claim</u>: $\frac{x_n}{x_{n+1}} \longrightarrow 1$.

<u>Need to show</u>: $\forall \gamma > 0, \exists N \in \mathbb{N}, \left|\frac{x_n}{x_{n+1}} - 1\right| < \gamma$ $\forall n \geq N$.

<u>We know</u>: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \left|\frac{1}{x_n}\right| < \epsilon$. $\forall n \geq N$.

~~Also know~~: $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, x_n \geq M \ \forall n \geq N$.

1. Consider $\gamma > 0$.
2. Using $M = 0$, in def. of $x_n \to \infty$, we get $N_1 \in \mathbb{N}$ s.t. $x_n \geq 0 \ \forall n \geq N_1$

Using $\epsilon = \gamma$, in def. of $\frac{1}{x_n} \to 0$,

we get $N_2 \in \mathbb{N}$ s.t. $\left|\frac{1}{x_n}\right| < \epsilon = \gamma$ $\forall n \geq N_2$.

Set $N = \max\{N_1, N_2\}$.

3. Consider $n \geq N$.

$$\left|\frac{x_n}{x_{n+1}} - 1\right| = \left|\frac{-1}{x_{n+1}}\right| = \left|\frac{1}{x_{n+1}}\right| < \left|\frac{1}{x_n}\right| \left(\text{since } x_n \geq 0\right)$$

and $\left|\frac{1}{x_n}\right| < \gamma.$ $\Rightarrow$ $\left|\frac{x_n}{x_{n+1}} - 1\right| < \gamma$ ▨.

4. <u>2.5.8.</u> Need to show: $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, s_n \geq M \ \forall n \geq N$

<u>Claim:</u> $s_n \geq \alpha^{n-1} s_1 \quad \forall n \in \mathbb{N}.$

Pf: By induction:

<u>Base case:</u> $s_1 = \alpha^{1-1} s_1$

<u>Induction step:</u> $s_{n-1} \geq \alpha^{n-2} s_1 \Rightarrow s_n \geq \alpha^{n-1} s_1$

~~Since~~ $s_{n-1} \geq \alpha^{n-2} s_1$

$\Rightarrow \alpha s_{n-1} \geq \alpha^{n-1} s_1$

$\Rightarrow s_n > \alpha s_{n-1} \geq \alpha^{n-1} s_1$ ▨.

1. Consider $M \in \mathbb{R}$.
2. Let N be a natural number $\geq \log_\alpha\left(\frac{M}{s_1}\right) + 1$
3. Consider $n \geq N$.

$$s_n \geq \alpha^{n-1} s_1$$

$$\geq \alpha^{N-1} s_1 \qquad \text{(since } \alpha > 1 \text{ and } n \geq N.)$$

$$\geq \alpha^{\log_\alpha \frac{M}{s_1}} s_1$$

$$= \frac{M}{s_1} \cdot s_1 = M$$ ▨.

5. 2.5.10 Need to show: $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geq N$
$s_n \geq M$.

1. Consider $M \in \mathbb{R}$.
2. For each $i \in \{1, \dots, p\}$.
   Let N_i be a natural number $\geq (2p|\alpha_i|)^{1/i}$

   Let N_0 be a natural number $\geq (2M)^{1/p}$
   Let $N = \max\{N_0, N_1, \dots, N_p\}$
3. Let $S = \{i \in \{1, \dots, p\} : \alpha_i < 0\}$ be the indices i such that α_i is negative.
   Consider $n \geq N$.
   If $i \in S$, we have

$$\frac{1}{n^i} \leq \frac{1}{(N_i)^i} \leq \frac{1}{2p|\alpha_i|}$$

$$\Rightarrow \frac{\alpha_i}{n^i} \geq \frac{\alpha_i}{2p|\alpha_i|} = \frac{-1}{2p} \qquad (1)$$

(since $i \in S \Rightarrow \alpha_i < 0$).

Now $n^p + \alpha_1 n^{p-1} + \alpha_2 n^{p-2} + \dots + \alpha_p$

$$= n^p\left(1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \dots + \frac{\alpha_p}{n^p}\right)$$

$$\geq n^p\left(1 + \sum_{i \in S} \frac{\alpha_i}{n^i}\right) \quad \text{[since we dropped nonnegative terms]}$$

$$\geq n^p\left(1 + \sum_{i \in S}\left(\frac{-1}{2p}\right)\right) \quad \text{By (1) above}$$

$$\geq n^p\left(1 + p\left(\frac{-1}{2p}\right)\right) \quad (\text{since } |S| \leq p).$$

$$\geq \frac{n^p}{2}$$

$$\geq \frac{N_0^p}{2} \qquad (\text{since } n \geq N_0).$$

$$= \frac{2M}{2} = M. \qquad \boxtimes$$

6. 2.6.1. a) False. Counterexample: $s_n = (-1)^n n$

b) ~~True~~ False. Counterexample $s_n = \begin{cases} 1 & n \text{ odd} \\ n & n \text{ even} \end{cases}$

~~Need to show~~

~~$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N},\ |s_n| \geq M\ \forall n \geq N$~~

c) True.

Pf: Let M_1 be such than $|s_n| \leq M_1$

Let M_2 be such than $|t_n| \leq M_2$.

Thus $|s_n + t_n| \leq |s_n| + |t_n| \leq M_1 + M_2$

$\therefore M_1 + M_2$ is a bound on $|s_n + t_n|$.

d). False. Counterexample $s_n = n$

$t_n = -n$.

e) True. Pf: Let M_1 be such that $|s_n| \leq M_1$

" M_2 " " " $|s_n| \leq M_2$.

Thus $|s_n t_n| = |s_n||t_n| \leq M_1 M_2$

$\Rightarrow M_1 M_2$ is a bound on $\{s_n t_n\}$.

f) False. Counterexample: $s_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ n & n \text{ even} \end{cases}$

$t_n = \begin{cases} n & n \text{ odd} \\ \frac{1}{n} & n \text{ even} \end{cases}$

$\{s_n t_n\} = \{1\}$

g) False. Counterexample $s_n = \frac{1}{n}$

2.6.2 Claim: $\lim \frac{s_n}{n} = 0$.

Pf: Consider $\epsilon > 0$.
Let M be a bound on $\{s_n\}$.
$\Rightarrow |s_n| \le M \quad \forall n \in \mathbb{N}$.
Choose N to be a natural number $> \frac{M}{\epsilon}$
Consider $n \geq N$

$$\left|\frac{s_n}{n} - 0\right| = \left|\frac{s_n}{n}\right|$$

$$= \frac{|s_n|}{n} \le \frac{M}{n} \quad \left(\text{since } |s_n| \le M \ \forall n \in \mathbb{N}\right)$$

$$\le \frac{M}{N} \quad (\text{since } n \geq N)$$

$$< \frac{M}{(M/\epsilon)} \quad \left(\text{since } N > \frac{M}{\epsilon}\right)$$

$$= \epsilon.$$

$\blacksquare$

2.6.3 False. Counterexample $s_n = (-1)^n$.

2.6.4. Contrapositive is true:

If $\{s_n\}$ is NOT bounded, the $\{s_n\}$ diverges.

2.6.5. Let $L = \lim s_n > 0$.

Letting $\epsilon = \frac{L}{2}$

we know $\exists N \in \mathbb{N}$ s.t. $|s_n - L| < \frac{L}{2}$ $\forall n \geq N$.

$\Rightarrow |L| - |s_n| < |s_n - L| < \frac{L}{2} \quad \forall n \geq N$

$\Rightarrow |L| - \frac{|L|}{2} < |s_n| = s_n \quad \forall n \geq N$.

$\Rightarrow \frac{|L|}{2} < s_n \quad \forall n \geq N$

Let $a = \min\{s_1, s_2, \dots, s_{N-1}\}$

Since each $s_1, s_2 \dots, s_{N-1}$ is positive,

$a > 0$.

Let $c = \min\{\frac{a}{2}, \frac{|L|}{2}\}$.

Claim: $s_n > c \quad \forall n \in \mathbb{N}$.

Pf: If $n < N$ then $s_n \geq a > \frac{a}{2} \geq c$.

If $n \geq N$ then $s_n > \frac{|L|}{2} \geq c$. ∎

7. Claim: $s_{2^n} \geq 1 + \frac{n}{2} \quad \forall n \geq \mathbb{N}$..

Pf: By induction.

Base case: $n=1$

$$LHS = S_{2^1} = S_2 = 1 + \frac{1}{2}$$

$$RHS = 1 + \frac{1}{2}$$

Thus $LHS \geq RHS$.

Induction step: Assume $S_{2^{n-1}} \geq 1 + \frac{n-1}{2}$

$$S_{2^n} = S_{2^{n-1}} + \underbrace{\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^n}}_{2^{n-1} \text{ terms}}$$

$$\geq S_{2^{n-1}} + \frac{2^{n-1}}{2^n}$$

Using Induction hypothesis

$$\geq 1 + \frac{n-1}{2} + \frac{1}{2} = 1 + \frac{n}{2}$$

The proof that S_n is unbounded follows:

Consider any $M \in \mathbb{R}$.

Let n be a natural no. $\geq 2(M)$.

Let $N = 2^n$.

$$S_N = S_{2^n} \geq 1 + \frac{n}{2} = 1 + \frac{2(M)}{2} = M+1 > M.$$

$\Rightarrow \{S_n\}$ is not bounded $\Rightarrow S_n$ diverges.

8. Since $s_n \to \infty$ we know:

$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $s_n \geq M \;\; \forall n \geq N$.

Claim: $\{s_n\}$ is not bounded.

Pf: Consider any $M \in \mathbb{R}$.

Using $M' = M+1$ in def. of $s_n \to \infty$,

we get $N \in \mathbb{N}$ s.t. $s_n \geq M' \;\; \forall n \geq N$

$\Rightarrow s_N \geq M' = M+1 > M$.

$\Rightarrow \{s_n\}$ is not bounded.

$\Rightarrow \{s_n\}$ diverges.

9. (i) • $\{s_n\}$ diverges.

Pf: We show $\{s_n\}$ is not bounded.

Consider $M \in \mathbb{R}$ and choose $n > \log_2 M$.

Let $N = 2n$.

$$s_N = 2^{2n/2} = 2^n > 2^{\log_2 M} = M.$$

• $\{s_n\}$ does not diverge to ∞.

Pf: Consider $M = 2$. For any $N \in \mathbb{N}$ consider

$n = 2N+1$.

Thus $n \geq N$ and $s_n = 1 < M$.

ii) • $\{s_n\}$ diverges to ∞.

Pf: $\forall M \in \mathbb{R}$, consider $N > M$. $s_n = n \geq N > M \ \forall n \geq N$.

~~$\Rightarrow \{s_n\}$ diverges ∞.~~

• $\{s_n\}$ diverges because $\{s_n\} \to \infty \Rightarrow \{s_n\}$ diverges.

iii) • $\{s_n\}$ diverges.

Pf: We show $\{s_n\}$ is not bounded.

Consider $M \in \mathbb{R}$.

Choose $n \in \mathbb{N}$ s.t. $n > M$ (Archimedean)

Let $N = 2n$.

$s_N = (-1)^N N = (-1)^{2n}(2n) = 2n > n > M$. ∎

• $\{s_n\}$ does not diverge to ∞.

Pf: Consider $M = 2$. For any $N \in \mathbb{N}$ consider $n = 2N+1$.

Thus, $n \geq N$ and $s_n = (-1)^{2N+1}(2N+1)$

$= -2N+1 < 0 < 2 = M$

∎

10. $\{s_n\}$ is convergent.

$\Rightarrow \{s_n \cdot s_n\} = \{s_n^2\}$ is convergent and.

$\lim s_n^2 = \lim s_n \cdot s_n = (\lim s_n)(\lim s_n) = (\lim s_n)^2$.

2.7.4. $(s_n - S)(t_n - T) + S(t_n - T) + T(s_n - S)$

$= s_n t_n - \cancel{S t_n} - \cancel{s_n T} + \cancel{ST} + \cancel{S t_n} - \cancel{ST} + \cancel{T s_n} - TS$

$= s_n t_n - ST$

$\Rightarrow |s_n t_n - ST| \leq |(s_n - S)(t_n - T)| + |S(t_n - T)| + |T(s_n - S)|$

Using $(|a+b| \leq |a| + |b|)$

Proof that $\{s_n t_n\} \longrightarrow ST$.

Consider $\gamma > 0$.

Using $\epsilon = \sqrt{\gamma/3}$ on def. of $s_n \to S$
get N_1 s.t. $|s_n - S| < \epsilon = \sqrt{\gamma/3}$ $\forall n \geq N_1$

Using $\epsilon = \sqrt{\gamma/3}$ on def of $t_n \to T$
get N_2 s.t. $|t_n - T| < \sqrt{\gamma/3}$ $\forall n \geq N_2$.

Using $\epsilon = \dfrac{\gamma}{3|S|}$ on def. of $t_n \to T$

get N_3 s.t $|t_n - T| < \dfrac{\gamma}{3|S|}$ $\forall n \geq N_3$.

Using $\epsilon = \dfrac{\gamma}{3|T|}$ on def. of $s_n \to S$.

get N_4 s.t. $|s_n - S| < \dfrac{\gamma}{3|T|}$ $\forall n \geq N_3$.

Let $N = \max\{N_1, N_2, N_3, N_4\}$.

Consider $n \geq N$

$$
\begin{aligned}
|s_n t_n - ST| &\leq |(s_n - S)(t_n - T)| + |S(t_n - T)| + |T(s_n - S)| \\
&\leq |s_n - S|\,|t_n - T| + |S|\,|t_n - T| + |T|\,|s_n - S| \\
&< \sqrt{\frac{\gamma}{3}}\sqrt{\frac{\gamma}{3}} + |S|\,|t_n - T| + |T|\,|s_n - S|) \\
&\qquad (\text{since } n \geq N_1 \text{ \& } n \geq N_2) \\
&< \frac{\gamma}{3} + |S|\frac{\gamma}{3|S|} + |T|\,|s_n - S| \\
&\qquad (\text{since } n \geq N_3) \\
&< \frac{\gamma}{3} + \frac{\gamma}{3} + |T|\frac{\gamma}{3|T|} \quad (\text{since } n \geq N_4) \\
&= \frac{\gamma}{3} + \frac{\gamma}{3} + \frac{\gamma}{3} = \gamma.
\end{aligned}
$$

∎

2.7.5 a) False $s_n = (-1)^n \quad \forall n \in \mathbb{N}$.

$t_n = (-1)^{n+1} \quad \forall n \in \mathbb{N}$

$\{s_n + t_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$

b) False. $s_n = (-1)^n$, $t_n = (-1)^n$.

$\{s_n t_n\} = \{1\}_{n=1}^{\infty}$

c) True. Since $\{s_n\}$ is convergent,

using $c = -1$,

$\{-s_n\}$ is convergent

by Theorem 2.14 in Text.

Since $\{-s_n\}$ and $\{s_n + t_n\}$ are both convergent

$\{(-s_n) + (s_n + t_n)\}$ is convergent

by Theorem 2.15 in text.

$\Rightarrow \{t_n\}$ is convergent.

d) False. $s_n = \frac{1}{n}$ $t_n = n$.

$\{s_n t_n\} = \{1\}_{n=1}^{\infty}$.

2.7.8. Theorem 2.15 and 2.16 only apply when $\{s_n\}$ and $\{t_n\}$ are convergent.

Instead we do the following manipulations:

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{(n+1) - (n)}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1/\sqrt{n}}{\frac{\sqrt{n+1}}{\sqrt{n}} + 1}$$

$$= \frac{1/\sqrt{n}}{\sqrt{1+\frac{1}{n}} + 1}.$$

Let $s_n = \frac{1}{\sqrt{n}}$ $\quad t_n = \sqrt{1+\frac{1}{n}} + 1$

We know $s_n \to 0$ also $\left\{1+\frac{1}{n}\right\} \to 1$

$\Rightarrow \sqrt{1+\frac{1}{n}} \to 1$ $\quad \Rightarrow t_n \to 1+1=2$

(by exercise 2.4.16).

Thus $\{s_n\}$ and $\{t_n\}$ both converge.

$$\Rightarrow \quad \lim \left(\sqrt{n+1} - \sqrt{n}\right)$$

$$= \lim \frac{1/\sqrt{n}}{\sqrt{1+\frac{1}{n}} + 1}$$

$$= \lim \frac{s_n}{t_n}$$

$$= \frac{\lim s_n}{\lim t_n} = \frac{0}{2} = 0.$$

Thus $\lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

For $\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

$$= \sqrt{n}\left(\frac{1/\sqrt{n}}{\sqrt{1+\frac{1}{n}} + 1}\right) = \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{t_n}.$$

$$\Rightarrow \quad \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim \frac{1}{t_n}$$

$$= \frac{\lim 1}{\lim t_n}$$

$$= \frac{1}{2}$$