HW 6 Solutions

1. Claim: $\lim_{n\to\infty} s_n = 0$.

Need to show: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \ |s_n - 0| < \epsilon$.

1. Consider $\epsilon > 0$.
2. Let N be a natural number larger than $\frac{\log \epsilon}{\log r} + 1$.
3. Consider $n \geq N$.

$$|s_n - 0| = |r^n - 0| = |r^n|$$

$$= r^n \leq r^N \quad \left(\text{since } r<1 \text{ and } n \geq N\right)$$

$$\leq r^{\frac{\log \epsilon}{\log r}+1}$$

$$= r \cdot r^{\log \epsilon / \log r}$$

$$< r^{\log \epsilon / \log r} \quad (\text{since } r < 1).$$

Since $\frac{\log \epsilon}{\log r} = \log_r \epsilon$, $r^{\frac{\log \epsilon}{\log r}} = r^{\log_r \epsilon} = \epsilon$.

$$\Rightarrow |s_n - 0| < r^{\log \epsilon / \log r} = \epsilon.$$

Scratch Work:

$r^n < \epsilon$

$\Rightarrow n \log r < \log \epsilon$

$\Rightarrow n > \frac{\log \epsilon}{\log r}$.

(since $\log r < 0$ because $r < 1$)

2. i) Claim : $\lim_{n\to\infty} e^{-n} = 0.$

<u>Need to show:</u> $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \; |s_n - L| \leq \epsilon.$

1. Consider $\epsilon > 0$.
2. Let N be a natural number larger than $-\log\epsilon + 1$. (by Archimedean principle).
3. Consider $n \geq N$.
4. $|s_n - 0| = |e^{-n}| = e^{-n}$

$$\leq e^{-N} \quad \left(\begin{matrix}\text{since} \\ n \geq N\end{matrix}\right)$$

$$\leq e^{-(-\log\epsilon+1)} \left(N \geq -\log\epsilon\right).$$

$$= e^{\log\epsilon} e^{-1} = \epsilon/e < \epsilon.$$

<u>Scratch Work</u>

$$|e^{-n} - 0| < \epsilon$$

$$\Rightarrow e^{-n} < \epsilon$$

$$\Rightarrow -n < \log\epsilon$$

$$\Rightarrow n > -\log\epsilon.$$

---

ii) <u>Claim</u>: $\lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$

Need to show: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \; |s_n - 1| < \epsilon.$

1. Consider $\epsilon > 0$.
2. Let N be a natural number larger than $\frac{\log 1/\epsilon}{\log 2} + 1$

<u>Scratch work</u>

$$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \epsilon$$

$$\Rightarrow \frac{1}{2^n} < \epsilon$$

$$\Rightarrow \frac{1}{\epsilon} < 2^n$$

3. Consider $n \geq N$.

4. $|s_n - 1| = \left|\left(1 - \frac{1}{2^n}\right) - 1\right| = \frac{1}{2^n}$

$\leq \frac{1}{2^N}$ (since $n \geq N$)

$\leq \frac{1}{2^{(\log 1/\epsilon / \log 2 + 1)}} = \frac{1}{2} \frac{1}{2^{(\log 1/\epsilon / \log 2)}}$

$= \frac{1}{2(1/\epsilon)} = \frac{\epsilon}{2} < \epsilon.$

$\Rightarrow \log \frac{1}{\epsilon} < n \log 2$

$\Rightarrow \frac{\log 1/\epsilon}{\log 2} < n$

$\Rightarrow |s_n - 1| < \epsilon.$ ∎

3. 2.4.8 Suppose $\{s_n\}$ converges.
Let $L = \lim_{n\to\infty} s_n$.

claim: $\lim_{n\to\infty} 2s_n = 2L$.

Pf: Since $\lim_{n\to\infty} s_n = L$, we have: $\forall \epsilon > 0, \exists N \in \mathbb{N}$, $|s_n - L| < \epsilon \ \forall n \geq N$.

We need to show: $\forall \gamma > 0, \exists N \in \mathbb{N}$, $|2s_n - 2L| < \gamma \ \forall n \geq N$.

1. Consider $\gamma > 0$.
2. Let $\epsilon = \gamma/2$. By def. of $\lim_{n\to\infty} s_n = L$, we get a natural number N' s.t. $|s_n - L| < \epsilon \ \forall n \geq N'$

Choose $N = N'$.

3. Consider any $n \geq N$.

4. $|2s_n - 2L| = 2|s_n - L|$

$< 2\epsilon$ (since $n \geq N = N'$ $\Rightarrow |s_n - L| < \epsilon$).

$= 2\frac{\gamma}{2}$

$= \gamma$.

$\Rightarrow |2s_n - 2L| < \gamma$. ■

2.4.12 We first show $\{s_n\}$ converges $\Rightarrow$ $\{t_n\}$ converges.

Since $\{s_n\}$ converges, say $\lim_{n\to\infty} s_n = L$.

Claim: $\lim_{n\to\infty} t_n = L$.

Pf: Since $\lim_{n\to\infty} s_n = L$, $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N\ |s_n - L| < \epsilon$.

Need to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, \forall n \geq N\ |t_n - L| < \gamma$

1. Consider $\gamma > 0$.
2. Let $\epsilon = \gamma$ and use def. of $\lim_{n\to\infty} s_n = L$ to get $N' \in \mathbb{N}$ s.t. $|s_n - L| < \epsilon\ \forall n \geq N'$.

Choose $N = N' = M$.

3. Consider $n \geq N$.

$$|t_n - L| = |s_{M+n} - L|$$

Since $n \geq N = N' - M$

$M + n \geq N'$

$\Rightarrow |s_{M+n} - L| < \epsilon$.

$\Rightarrow |t_n - L| < \epsilon = \gamma$.

We now show: $\{t_n\}$ converges $\Rightarrow$ $\{s_n\}$ converges.

~~Consider~~ Since $\{t_n\}$ converges, say $\lim_{n\to\infty} t_n = L$

<u>Claim</u>: $\lim_{n\to\infty} s_n = L$.

Pf: Since $\lim_{n\to\infty} t_n = L$, $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \; |t_n - L| < \epsilon$.

Need to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, \forall n \geq N \; |t_n - L| < \gamma$.

1. Consider $\gamma > 0$.
2. Let $\epsilon = \gamma$, and use def. of $\lim_{n\to\infty} t_n = L$ to get $N' \in \mathbb{N}$ s.t. $|t_n - L| < \epsilon \;\; \forall n \geq N'$.

Choose $N = N' + M$.

3. Consider $n \geq N$.
4. Since $n \geq N = N' + M$

   $n - M \geq N'$

Since $s_n = t_{n-M} \Rightarrow |s_n - L| = |t_{n-M} - L| < \epsilon = \gamma$.

$\Rightarrow |s_n - L| < \gamma$.

2.4.15 $\{s_n\} \to 0 \Rightarrow \{\sqrt{s_n}\} \to 0.$

$\lim_{n\to\infty} s_n = 0 \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \ |s_n - 0| < \epsilon.$

Need to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, \forall n \geq N \ |\sqrt{s_n} - 0| < \gamma$

1. Consider any $\gamma > 0$.
2. Let $\epsilon = \gamma^2$ and use def. of $\lim_{n\to\infty} s_n = 0$ to get $N' \in \mathbb{N}$ s.t. $|s_n| < \epsilon = \gamma^2 \ \forall n \geq N'$.

   Choose $N = N'$.
3. Consider $n \geq N$
4. $|\sqrt{s_n} - 0| = |\sqrt{s_n}| = \sqrt{s_n} < \sqrt{\gamma^2}$ (since $n \geq N = N'$, $|s_n| = s_n < \gamma^2$)

   $= \gamma.$

   $\Rightarrow |\sqrt{s_n} - 0| < \gamma.$ ∎.

2.4.16. $\{s_n\} \to L \Rightarrow (\{\sqrt{s_n}\} \to \sqrt{L})$

$\lim_{n\to\infty} s_n = L \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \ |s_n - L| < \epsilon.$

Need to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, \forall n \geq N. \ |\sqrt{s_n} - \sqrt{L}| < \gamma.$

1. Consider $\gamma > 0$.
2. Let $\epsilon = \gamma\sqrt{L}$ and use def. of $\lim_{n\to\infty} s_n = L$ to get $N' \in \mathbb{N}$ s.t. $|s_n - L| < \epsilon \ \forall n \geq N'$.

   Choose $N = N'$.
3. Consider $n \geq N$.
4. $|\sqrt{s_n} - \sqrt{L}| \leq \left|\frac{s_n - L}{\sqrt{s_n} + \sqrt{L}}\right|$

$$\leq \frac{|s_n - L|}{\sqrt{s_n}+\sqrt{L}}$$

$$< \frac{\epsilon}{\sqrt{s_n}+\sqrt{L}} \qquad (\text{since } n \geq N = N')$$

$$\leq \frac{\epsilon}{\sqrt{L}} \qquad \left(\text{since } \sqrt{s_n} \geq 0 \Rightarrow \frac{1}{\sqrt{s_n}+\sqrt{L}} \leq \frac{1}{\sqrt{L}}\right)$$

$$= \frac{\gamma\sqrt{L}}{\sqrt{L}} = \gamma.$$

$\blacksquare$

2.4.9. To show $s_n = n$ diverges, we need to show:

$$\forall L \in \mathbb{R}, \quad \lim_{n\to\infty} s_n \neq L.$$

1. Consider an arbitrary $L \in \mathbb{R}$.
2. Let $\epsilon = 1$.
3. Consider arbitrary $N \in \mathbb{N}$.
4. Choose $n = \max\{N, L+1\}$.
5. $n \geq N$. since $n = \max\{N, L+1\}$.

$$|s_n - L| = |n - L|$$

$$\geq n - L \geq L+1-L \quad (\text{since } n \geq L+1)$$

$$= 1 = \epsilon.$$

$\blacksquare$

2.4.10 to show $s_n = (-1)^n$ diverges, need to show.

$$\forall L \in \mathbb{R}, \lim_{n\to\infty} s_n \neq L.$$

1. Consider an arbitrary $L \in \mathbb{R}$.
2. Let $\epsilon = 1/2$
3. Consider arbitrary $N \in \mathbb{N}$.
4. Choose $n = 2N+1$ if $L \geq 0$

   and $n = 2N$ if $L < 0$.
5. Either $n = 2N \geq N$

   or $n = 2N+1 \geq N$

   thus in both cases, $n \geq N$.

if $L \geq 0$

$$|s_n - L| = |(-1)^{2N+1} - L| = |-1-L| = 1+L \geq 1 \geq 1/2.$$

if $L<0$

$$|s_n - L| = |(-1)^{2N} - L| = |1-L| = 1-L \geq 1 \geq 1/2.$$

In both cases, $|s_n - L| \geq 1/2 = \epsilon$. ∎

5. $\{s_n\}_{n=1}^{\infty}$ is a sequence with $s_n \in \mathbb{Z}$ $\forall n \in \mathbb{N}$.

$L = \lim_{n\to\infty} s_n$. Need to show $L \in \mathbb{Z}$.

By contradiction: Assume $L \notin \mathbb{Z}$.

By Problem 2, HW 5,

$\exists\, m \in \mathbb{Z}$ s.t.

$$m-1 \leq L < m.$$

Since $L \notin \mathbb{Z}$ and $m-1 \in \mathbb{Z}$

we actually have $m-1 < L < m$.

We now show $\lim_{n\to\infty} s_n \neq L$ reaching a contradiction.

1. Let $\epsilon = \min\left\{ \frac{m-L}{2}, \frac{L-(m-1)}{2} \right\}$.
   Since $m-1 < L < m$, $\epsilon > 0$.

2. Consider arbitrary $N \in \mathbb{N}$.

3. Let $n = N$.

4. Clearly $n \geq N$. We analyze $|s_n - L|$.
   Since $s_n \in \mathbb{Z}$, we have
   $s_n \geq m$ or $s_n \leq m-1$.

Case 1: $s_n \geq m$.

$$|s_n - L| \geq s_n - L \geq m - L > \frac{m-L}{2} \geq \epsilon.$$

Case 2: $s_n \leq m-1$

$$|s_n - L| \geq L - s_n \geq L-(m-1) \qquad \left(\text{since } s_n \leq m-1\right)$$

$$> \frac{L-(m-1)}{2} \geq \epsilon.$$

In both cases $|s_n - L| \geq \epsilon$.

---

6. (i) $s_n = (-1)^n$ does not converge (diverges).

Pf: This is 2.4.10.

(ii) $s_n = \frac{(-1)^n}{n}$ converges to 0.

Pf: Need to show $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|s_n - 0| < \epsilon$.

Consider $\epsilon > 0$. Pick N larger than $1/\epsilon + 1$

For any $n \geq N$, $|s_n - 0| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \leq \frac{1}{N} \leq \frac{1}{(1/\epsilon + 1)}$

$$< \frac{1}{1/\epsilon} = \epsilon.$$

(iii) $s_n = (-1)^n \cdot 2$ diverges.

Pf: Almost the same proof as 2.4.10.

Can choose any $\epsilon \leq 2$.

(iv) $s_n = \frac{3n^3+1}{4n^3+2}$ converges to $\frac{3}{4}$.

Pf:

1. Consider any $\epsilon > 0$.

2. Choose $N \in \mathbb{N}$ larger than

$$\frac{1}{2\sqrt[3]{\epsilon}}.$$

3. Consider $n \geq N$.

4. $$|s_n - L| = \left|\frac{3n^3+1}{4n^3+2} - \frac{3}{4}\right|$$

$$= \left|\frac{-2}{4(4n^3+2)}\right|$$

$$= \frac{1}{2(4n^3+2)}.$$

Now $n \geq N \geq \dfrac{1}{2\sqrt[3]{\epsilon}}$

$$\Rightarrow n^3 \geq \frac{1}{8\epsilon}$$

$$\Rightarrow 4n^3 \geq \frac{1}{2\epsilon} \Rightarrow 4n^3+2 > \frac{1}{2\epsilon}$$

$$\Rightarrow \epsilon > \frac{1}{2(4n^3+2)}.$$

$$\Rightarrow |s_n - L| = \frac{1}{2(4n^3+2)} < \epsilon.$$

Scratch work

$$\left|\frac{3n^3+1}{4n^3+2} - \frac{3}{4}\right| < \epsilon$$

$$\left|\frac{-2}{4(4n^3+2)}\right| < \epsilon$$

$$\frac{1}{2(4n^3+2)} < \epsilon$$

$$\frac{1}{2\epsilon} < 4n^3 + 2$$

$$\frac{1}{2\epsilon} - 2 < 4n^3$$

$$\frac{1}{8\epsilon} - \frac{1}{2} < n^3$$

Select $N \geq \dfrac{1}{2\sqrt[3]{\epsilon}}$